

Einstein-Maxwell-Massive Scalar Field System in 3+1 formulation on Bianchi Spacetimes type I-VIII

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Abstract

Global existence to the coupled Einstein-Maxwell-Massive Scalar Field system which rules the dynamics of a kind of charged pure matter in the presence of a massive scalar field is proved, in Bianchi I-VIII spacetimes; asymptotic behaviour, geodesic completeness, energy conditions are investigated in the case of a cosmological constant bounded from below by a strictly negative constant depending only on the massive scalar field.

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Introduction

In relativistic kinetic theory, global dynamics of several kinds of charged and uncharged matter remain an active research domain in General Relativity (GR), in which cosmology plays one of the central roles, by coupling various matter fields to the Einstein equations to provide mathematical explanations

in response to new astrophysical observations. In this context, spatially homogeneous phenomena such as the one we consider in the present paper are relevant. There are several reasons why it is of interest to consider the Einstein equations with cosmological constant and to couple the equations to a massive scalar field.

- Astrophysical observations have made evident the fact that, even in the presence of material bodies, the gravitational field can propagate through space at the speed of the light, analogously to electromagnetic waves. A mathematical way to model this phenomenon is to couple a scalar field to the Einstein equations. Let us recall that the Nobel prize of Physics 1993 was awarded for works on this subject. More details on this question can be found in [5], [21]. In fact, several authors realized the interest of coupling scalar field to other fields equations; see for instance [4], [18], [20], [24], [8].

- Now our motivation for considering the Einstein equations with a cosmological constant Λ is due to the fact that astrophysical observations, based on luminosity via redshift plots of some far away objets such as Supernovae-Ia, have made evident the fact that the expansion of the universe is accelerating, as foreseen by E.P. Hubble. A classical mathematical tool to model this phenomenon is to include the cosmological constant Λ in the Einstein Equations. Several authors did it in the case $\Lambda > 0$; see for instance [14], [7], [15], [22], [23], [26]. In the present paper, we prove that, in the presence of a massive scalar field, this result can be extended, not only to the case $\Lambda = 0$, but also to the case $\Lambda > -\alpha^2$, where $\alpha > 0$ is a constant depending only on the potential of the massive scalar field. This result extends and completes those of [20].

Also recall that the recent Nobel prize of Physics, 2011, was awarded to three Astrophysicists for their advanced research on this phenomenon of accelerated expansion of the universe.

In fact, we must point out that, the notion of "dark energy" was introduced in order to provide a physical explanation to this phenomenon, but the physical structure of this hypothetical form of energy which is unknown in the laboratories remains an open question in modern cosmology; so is the question of "dark matter". Also notice that the scalar fields are considered to be a mechanism producing accelerated models, non only in "inflation", which is a variant of the Big-Bang theory including now a very short period of very high acceleration, but also in the primordial universe.

In this paper, we consider the 3+1 formulation of the Einstein equations, which allows to interpret the fields equations as the time history of the first and second fundamental forms of the 3-hypersurfaces of constant times slices, foliating the space-time. The background space-time is any Bianchi space-time type I to VIII, since it is proved in [17] that Bianchi IX such as the Kantowski-Sachs space-time, develops curvature singularities in a finite proper time, and this constitutes a major obstacle to our goal which is to proved global in time

existence of solutions. We prove in this paper that if the initial value of the mean curvature is strictly negative and if $\Lambda > -\alpha^2$, then the coupled Einstein-Maxwell-Scalar Field System has a global in time-solution. We investigate the asymptotic behaviour which reveals an exponential growth of the gravitational potentials, confirming the accelerated expansion of the universe. We prove the geodesic completeness and we were able to show that the considered model always satisfies the weak and the dominant energy conditions; we prove that, if $\Lambda > \beta^2$ where $\beta > 0$ is a constant depending only on the mean curvature of the space-time, then the considered model also satisfies the strong energy condition.

The paper is organized as follows:

- In section 1, we introduce the coupled system and we give some preliminary results.
- In section 2, we study the constraints equations, the mean curvature and we introduce the Cauchy problem.
- In section 3, we prove the local and the global existence of solutions.
- In section 4, we study the asymptotic behaviour.
- In section 5, we study the geodesic completeness.
- In section 6, we study the energy conditions.
- Section 7 is the appendix to which we refer for the details of the proofs of some important results.

1 Equations and preliminary results

- Unless otherwise specified, Greek indices $\alpha, \beta, \lambda, \dots$, range from 0 to 3 and Latin indices i, j, k, \dots , from 1 to 3. We adopt the Einstein summation convention $a_\alpha b^\alpha \equiv \sum_\alpha a_\alpha b^\alpha$.

We consider a time-oriented space-time (M, \tilde{g}) where M is a four-dimensional manifold and \tilde{g} the metric tensor of lorentzian signature $(-, +, +, +)$. The model adopted for our study is any Bianchi space-time from type I to VIII, and, following [17], [19], [25], [27], who studied the question, we take M on the form: $M = \mathbb{R} \times G$, where G is a three dimensional simply connected Lie group. We take \tilde{g} on the form:

$$\tilde{g} = -dt^2 + g_{ij}(t)e^i \otimes e^j \quad (1)$$

where $\{e_i\}$ is a left invariant frame on G and $\{e^i\}$ is the dual frame; (g_{ij}) is a positive definite Riemannian 3-metric depending only on t . Now the

vector $n = \partial_t$ being orthogonal to G , we complete the frame (e_i) on G , to obtain a frame (n, e_i) on M . We have

$$\tilde{g}(n, e_i) = \tilde{g}_{0i} = 0,$$

confirming the form (1) of \tilde{g} .

- The Einstein-Maxwell-Massive scalar field system with cosmological constant Λ , which rules the evolution of the considered charged pure matter can be written, following [9], and denoting with a tilda (\sim) quantities on M :

$$\left\{ \begin{array}{l} \tilde{R}_{\alpha\beta} - \frac{1}{2}\tilde{R}\tilde{g}_{\alpha\beta} + \Lambda\tilde{g}_{\alpha\beta} = 8\pi(\tilde{T}_{\alpha\beta} + \tilde{\tau}_{\alpha\beta} + \rho\tilde{u}_\alpha\tilde{u}_\beta) \end{array} \right. \quad (2)$$

$$\left\{ \begin{array}{l} \tilde{\nabla}_\alpha \tilde{F}^{\alpha\beta} = 4\pi \tilde{J}^\beta \end{array} \right. \quad (3)$$

$$\left\{ \begin{array}{l} \tilde{\nabla}_\alpha \tilde{F}_{\beta\gamma} + \tilde{\nabla}_\beta \tilde{F}_{\gamma\alpha} + \tilde{\nabla}_\gamma \tilde{F}_{\alpha\beta} = 0 \end{array} \right. \quad (4)$$

where:

- (2) are the Einstein equations, basic equations in GR for the unknown metric tensor $\tilde{g} = (\tilde{g}_{\alpha\beta})$; $\tilde{R}_{\alpha\beta}$ is the Ricci tensor, contracted of the curvature tensor, $\tilde{R} = \tilde{g}^{\alpha\beta}\tilde{R}_{\alpha\beta}$ is the scalar curvature; $\tilde{T}_{\alpha\beta}$ and $\tilde{\tau}_{\alpha\beta}$ whose expressions are given below, are respectively, the tensor associated to a massive scalar field Φ which is an unknown function of the time t , and $\tilde{\tau}_{\alpha\beta}$ the Maxwell tensor associated to the electromagnetic field \tilde{F} .
- (3) and (4) are the two sets of Maxwell equations, basic equations of Electromagnetism, written in covariant form for the electromagnetic field $\tilde{F} = (\tilde{F}^{0i}, \tilde{F}_{ij})$ which is a closed unknown antisymmetric 2-form, depending only on the time t . \tilde{F}^{0i} and \tilde{F}_{ij} are respectively its electric and magnetic parts. $\tilde{T}_{\alpha\beta}$ and $\tilde{\tau}_{\alpha\beta}$ are defined by:

$$\left\{ \begin{array}{l} \tilde{T}_{\alpha\beta} = \tilde{\nabla}_\alpha \Phi \tilde{\nabla}_\beta \Phi - \frac{1}{2}\tilde{g}_{\alpha\beta} [\tilde{\nabla}^\lambda \Phi \tilde{\nabla}_\lambda \Phi + m^2 \Phi^2] \end{array} \right. \quad (5)$$

$$\left\{ \begin{array}{l} \tilde{\tau}_{\alpha\beta} = -\frac{1}{4}\tilde{g}_{\alpha\beta} \tilde{F}^{\lambda\mu} \tilde{F}_{\lambda\mu} + \tilde{F}_{\alpha\lambda} \tilde{F}_\beta{}^\lambda \end{array} \right. \quad (6)$$

where in (5) as in (3) and (4), $\tilde{\nabla}$ stands for the covariant derivative, or the Levi-Civita connection in \tilde{g} , $m > 0$ is a given constant called the *mass* of the scalar field Φ ; notice that $\frac{1}{2}m^2\Phi^2$ represents the potential associated to the scalar field Φ .

- In (2), $\rho\tilde{u}_\alpha\tilde{u}_\beta$ is the tensor associated to the considered charged pure matter, with $\rho > 0$ an unknown scalar function of the time t , standing for the pure matter proper density, and $\tilde{u} = (\tilde{u}^\beta)$ a time-like future pointing unit vector which is an unknown function of the time t , representing the material velocities.

- In (3), $\tilde{J} = (\tilde{J}^\beta)$ stands for the Maxwell current generated by the charged particles of pure matter and defined by:

$$\tilde{J}^\beta = e\tilde{u}^\beta \quad (7)$$

in which $e \geq 0$ is an unknown scalar function of the time t , standing for the proper charged density of the charged particles.

Notice that the Maxwell equations (4) are just the covariant notation of the relation $d\tilde{F} = 0$, since \tilde{F} is a closed 2-form.

Now it is well known, see [12], that the electromagnetic field deviates the trajectories of the charged particles which are no longer the geodesics of the space-time as in the empty case, but the solutions of the following differential system of current flow:

$$\tilde{u}^\alpha \tilde{\nabla}_\alpha \tilde{u}^\beta = 4\pi \frac{e}{\rho} \tilde{F}_\lambda{}^\beta \tilde{u}^\lambda \quad (8)$$

which reduces to the usual geodesics system when $\tilde{F} = 0$.

- To study the Einstein equations (2), we adopt the 3+1 formulation, which allows to interpret these equations as the evolution in time of the triplet (Σ_t, g_t, k_t) , where $\Sigma_t = \{t\} \times G$, $g_t = (g_{ij}(t))$ stands for the first fundamental form induced on Σ_t by \tilde{g} , $k_t = (k_{ij}(t))$ is the second fundamental form defined in the present case by:

$$k_{ij} = -\frac{1}{2}\partial_t g_{ij} \quad (9)$$

where we adopt the ADM/MTV convention see [1], [6].

By the 3+1 formulation, the Einstein equations (2) which originally, in the homogeneous case we consider, are a non linear *second order* differential system in g_{ij} , can be written as an equivalent *first order* differential system in (g_{ij}, k_{ij}) to which standard theory applies.

We now introduce a quantity which will play a central role, namely, the *mean curvature* of the space-time which is a scalar function denoted by H and defined by:

$$H = g^{ij} k_{ij} \quad (10)$$

that is the trace of (k_{ij}) . Next, since $\tilde{u} = (\tilde{u}^\beta)$ is a unit vector which means $\tilde{g}(\tilde{u}, \tilde{u}) = \tilde{u}_\alpha \tilde{u}^\alpha = -1$, we deduce from the expression (1) of \tilde{g} that:

$$\tilde{u}^0 = \sqrt{1 + g_{ij} \tilde{u}^i \tilde{u}^j} \quad (11)$$

because \tilde{u} is future pointing, and (11) shows that g_{ij} and u^i determine \tilde{u}^0 , and, since (g_{ij}) is positive definite, that $\tilde{u}^0 \geq 1$.

- It is important to note that (e_i) is not always the natural frame (∂_i) on G . We set $(\tilde{e}_\alpha) = (n, e_i) = (\partial_t, e_i)$: that is $\tilde{e}_0 = \partial_t$ and $\tilde{e}_i = e_i$. The frames (\tilde{e}_α) and (∂_α) are then link by:

$$\tilde{e}_\alpha = \tilde{e}_\alpha^\lambda \partial_\lambda \quad (12)$$

where

$$\tilde{e}_0^0 = 1; \quad \tilde{e}_i^0 = \tilde{e}_0^i = 0; \quad \tilde{e}_j^i = e_j^i. \quad (13)$$

Recall that the structure constants C_{ij}^k of the Lie algebra \mathfrak{g} of the Lie group G are defined by:

$$[e_i, e_j] = C_{ij}^k e_k \quad (14)$$

where $[,]$ denotes the Lie brackets of \mathfrak{g} . Since the Lie brackets are antisymmetric, C_{ij}^k is antisymmetric with respect to i, j that is:

$$C_{ij}^k = -C_{ji}^k. \quad (15)$$

Now the Ricci rotation coefficients $\tilde{\gamma}_{\alpha\beta}^\lambda$ associated to the Levi-Civita connection $\tilde{\nabla}$ are defined by:

$$\tilde{\nabla}_{\tilde{e}_\alpha} \tilde{e}_\beta = \tilde{\gamma}_{\alpha\beta}^\lambda \tilde{e}_\lambda. \quad (16)$$

We set

$$[\tilde{e}_\alpha, \tilde{e}_\beta] = \tilde{C}_{\alpha\beta}^\lambda \tilde{e}_\lambda \quad (17)$$

where:

$$\tilde{C}_{\alpha\beta}^0 = \tilde{C}_{0\beta}^\lambda = \tilde{C}_{\alpha 0}^\lambda = 0; \quad \tilde{C}_{ij}^k = C_{ij}^k. \quad (18)$$

The general expression of $\tilde{\gamma}_{\alpha\beta}^\lambda$ in term of $\tilde{C}_{\alpha\beta}^\lambda$ and $\tilde{g}_{\alpha\beta}$ can be find in [2] or in [3] p.301. It will be enough to extract and mention here only those of the Ricci rotation coefficients used in the present paper, namely:

$$\begin{cases} \tilde{\gamma}_{00}^\lambda = \tilde{\gamma}_{i0}^\lambda = \tilde{\gamma}_{0i}^\lambda = 0; \quad \tilde{\gamma}_{ij}^0 = -k_{ij}; \quad \tilde{\gamma}_{i0}^j = \tilde{\gamma}_{0i}^j = -k_i^j \\ \tilde{\gamma}_{ij}^l := \gamma_{ij}^l = \frac{1}{2} g^{lk} \left[-C_{jk}^m g_{im} + C_{ki}^m g_{jm} + C_{ij}^m g_{km} \right]. \end{cases} \quad (19)$$

Notice that the $\tilde{\gamma}_{\alpha\beta}^\lambda$ are not to be confused with the usual Christoffel symbols $\Gamma_{\alpha\beta}^\lambda$ associated to the natural frame (∂_λ) . For instance, (19) gives, using $C_{ij}^k = -C_{ji}^k$:

$$\gamma_{ij}^i = C_{ij}^i; \quad \gamma_{ij}^l - \gamma_{ji}^l = C_{ij}^l \quad (20)$$

which shows that, at the contrary of the $\Gamma_{\alpha\beta}^\lambda$, the $\tilde{\gamma}_{\alpha\beta}^\lambda$ are not symmetric with respect to α and β . We have the following formulae, analogous to the natural frame case:

$$\begin{cases} \tilde{\nabla}_\alpha \tilde{T}^\beta = \tilde{e}_\alpha(\tilde{T}^\beta) + \tilde{\gamma}_{\alpha\lambda}^\beta \tilde{T}^\lambda \\ \tilde{\nabla}_\alpha \tilde{T}_\beta = \tilde{e}_\alpha(\tilde{T}_\beta) - \tilde{\gamma}_{\alpha\beta}^\lambda \tilde{T}_\lambda. \end{cases} \quad (21)$$

Also mention the useful formula we prove in Appendix A₁:

$$\frac{d}{dt}(\gamma_{ij}^l) = \nabla^l k_{ij} - \nabla_j k_i^l - \nabla_i k_j^l \quad (22)$$

where ∇ is the Levi-Civita connection on (G, g) .

- A direct calculation using (21) shows that the components of the tensor $\tilde{T}_{\alpha\beta}$ defined by (5) are given, setting $\dot{\Phi} = \frac{d\Phi}{dt}$, by:

$$\tilde{T}_{00} = \frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}m^2\Phi^2; \quad \tilde{T}_{0i} = 0; \quad \tilde{T}_{ij} = \frac{1}{2}g_{ij}(\dot{\Phi}^2 - m^2\Phi^2). \quad (23)$$

Concerning the Maxwell tensor $\tilde{\tau}_{\alpha\beta}$ defined by (6), we first obtain by a direct calculation:

$$\begin{cases} \tilde{F}^{\lambda\mu}\tilde{F}_{\lambda\mu} = -2g_{ij}\tilde{F}^{0i}\tilde{F}^{0j} + g^{ik}g^{jl}\tilde{F}_{ij}\tilde{F}_{kl}; \\ \tilde{F}_{i\lambda}\tilde{F}_j{}^\lambda = -g_{ik}g_{jl}\tilde{F}^{0k}\tilde{F}^{0l} + g^{kl}\tilde{F}_{ik}\tilde{F}_{jl}; \\ \tilde{F}_{0\lambda}\tilde{F}_j{}^\lambda = -\tilde{F}_{jk}\tilde{F}^{0k}; \\ \tilde{F}_{0\lambda}\tilde{F}_0{}^\lambda = g_{ij}\tilde{F}^{0i}\tilde{F}^{0j}; \end{cases} \quad (24)$$

from where we deduce, using (6), that:

$$\begin{cases} \tilde{\tau}_{00} = \frac{1}{2}g_{ij}\tilde{F}^{0i}\tilde{F}^{0j} + \frac{1}{4}g^{ik}g^{jl}\tilde{F}_{kl}\tilde{F}_{ij} \\ \tilde{\tau}_{0j} = -\tilde{F}^{0k}\tilde{F}_{jk} \\ \tilde{\tau}_{ij} = (\frac{1}{2}g_{ij}g_{kl} - g_{ik}g_{jl})\tilde{F}^{0k}\tilde{F}^{0l} - \frac{1}{4}g_{ij}g^{km}g^{nl}\tilde{F}_{kl}\tilde{F}_{mn} + g^{kl}\tilde{F}_{ik}\tilde{F}_{jl}. \end{cases} \quad (25)$$

- Next, to derive the equations for the scalar field Φ and the matter density ρ , we use the conservation laws:

$$\widetilde{\nabla}_\alpha \tilde{T}^{\alpha\beta} + \widetilde{\nabla}_\alpha \tilde{\tau}^{\alpha\beta} + \widetilde{\nabla}_\alpha (\rho \tilde{u}^\alpha \tilde{u}^\beta) = 0. \quad (26)$$

A direct calculation using (5), (6), (21) and the Maxwell equation (4) gives:

$$\widetilde{\nabla}_\alpha \tilde{T}^{\alpha\beta} = \widetilde{\nabla}^\beta \Phi (\widetilde{\nabla}_\alpha \widetilde{\nabla}^\alpha \Phi - m^2 \Phi); \quad \widetilde{\nabla}_\alpha \tilde{\tau}^{\alpha\beta} = \tilde{F}^\beta{}_\lambda \widetilde{\nabla}_\alpha \tilde{F}^{\alpha\lambda} \quad (27)$$

where $\widetilde{\nabla}_\alpha \widetilde{\nabla}^\alpha$ often denoted $\square_{\tilde{g}}$ is the d'Alembertian or the wave operator. Now (26) and (27) give, using the Maxwell equation (3):

$$\widetilde{\nabla}^\beta \Phi (\widetilde{\nabla}_\alpha \widetilde{\nabla}^\alpha \Phi - m^2 \Phi) + 4\pi e \tilde{F}^\beta{}_\lambda \tilde{u}^\lambda + \tilde{u}^\beta \widetilde{\nabla}_\alpha (\rho \tilde{u}^\alpha) + (\rho \tilde{u}^\alpha) \widetilde{\nabla}_\alpha \tilde{u}^\beta = 0 \quad (28)$$

(28), reduces using (8) to:

$$\widetilde{\nabla}^\beta \Phi (\widetilde{\nabla}_\alpha \widetilde{\nabla}^\alpha \Phi - m^2 \Phi) + \tilde{u}^\beta \widetilde{\nabla}_\alpha (\rho \tilde{u}^\alpha) = 0. \quad (29)$$

But it is easily seen that $\widetilde{\nabla}^i \Phi = 0$; so (29) gives for $\beta = i$, $\widetilde{u}^i \widetilde{\nabla}_\alpha (\rho \widetilde{u}^\alpha) = 0$ which is satisfied for every $i = 1, 2, 3$ if:

$$\widetilde{\nabla}_\alpha (\rho \widetilde{u}^\alpha) = 0. \quad (30)$$

Now (29) gives, since $\widetilde{\nabla}^i \Phi = 0$ and using (30):

$$\widetilde{\nabla}^0 \Phi (\widetilde{\nabla}_\alpha \widetilde{\nabla}^\alpha \Phi - m^2 \Phi) = 0. \quad (31)$$

It then appears that the conservation laws (26) will be satisfied if equations (30) and (31) in ρ and Φ are.

- Now a direct calculation shows that equation (30) in ρ writes:

$$\dot{\rho} = - \left(\frac{\dot{\widetilde{u}}^0}{\widetilde{u}^0} - k_i^i + C_{ij}^i \frac{\widetilde{u}^j}{\widetilde{u}^0} \right) \rho,$$

which solves at once over $[0, t]$, $t > 0$, to give:

$$\rho = \frac{\rho(0) \widetilde{u}^0(0)}{\widetilde{u}^0} \exp \left[\int_0^t (k_i^i - C_{ij}^i \frac{\widetilde{u}^j}{\widetilde{u}^0})(s) ds \right]. \quad (32)$$

(32) shows that $(\rho(0) > 0) \implies (\rho > 0)$. In what follows we set:

$$\rho(0) > 0. \quad (33)$$

Next, it is easily seen, using (21), that

$$\widetilde{\nabla}_\alpha \widetilde{\nabla}^\alpha \Phi = -\ddot{\Phi} + H\dot{\Phi},$$

so that equation (31) in Φ can be written, using $\widetilde{\nabla}^0 \Phi = \widetilde{g}^{00} \widetilde{\nabla}_0 \Phi = -\dot{\Phi}$:

$$\dot{\Phi} \ddot{\Phi} - H(\dot{\Phi})^2 + m^2 \Phi \dot{\Phi} = 0. \quad (34)$$

To study this non linear second order equation in Φ , we set:

$$U = \frac{1}{2}(\dot{\Phi})^2. \quad (35)$$

We choose to look for a non-decreasing scalar field Φ , which means $\dot{\Phi} \geq 0$; (35) then gives:

$$\dot{\Phi} = \sqrt{2U}. \quad (36)$$

- Next, to derive equation in e , we use the Maxwell equation (3), which gives, using the identity $\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \widetilde{F}^{\alpha\beta} = 0$ and the expression (7) of \widetilde{J}^β :

$$\widetilde{\nabla}_\alpha (e \widetilde{u}^\alpha) = 0 \quad (37)$$

(37), yields, similarly to (30):

$$e = \frac{e(0)\tilde{u}^0(0)}{\tilde{u}^0} \exp \left[\int_0^t (k_i^i - C_{ij}^i \frac{\tilde{u}^j}{\tilde{u}^0})(s) ds \right] \quad (38)$$

by (38): $(e(0) \geq 0) \implies e \geq 0$. In what follows we set:

$$e(0) \geq 0. \quad (39)$$

- We now write in details the Maxwell equations (3), (4) in \tilde{F} ; (3) gives for $\beta = i$ and $\beta = 0$, and using the expression (7) of \tilde{J}^β

$$\begin{cases} \tilde{\nabla}_\alpha \tilde{F}^{\alpha i} = 4\pi e \tilde{u}^i \\ \tilde{\nabla}_\alpha \tilde{F}^{\alpha 0} = 4\pi e \tilde{u}^0. \end{cases} \quad (40)$$

$$\quad (41)$$

- Applying formulae (21) yields:

$$\tilde{\nabla}_\alpha \tilde{F}^{\alpha i} = \tilde{e}_\alpha \tilde{F}^{\alpha i} + \tilde{\gamma}_{\alpha\lambda}^\alpha \tilde{F}^{\lambda i} + \tilde{\gamma}_{\alpha\lambda}^i \tilde{F}^{\alpha\lambda}. \quad (42)$$

But by a direct calculation using (19), (20) the properties $C_{ij}^k = -C_{ji}^k$, $\tilde{F}^{ij} = -\tilde{F}^{ji}$ and since

$$\tilde{e}_\alpha \tilde{F}^{\alpha i} = \partial_0 \tilde{F}^{0i} = \dot{\tilde{F}}^{0i}$$

we deduce from (40) and (42), the equation in \tilde{F}^{0i} :

$$\dot{\tilde{F}}^{0i} = H \tilde{F}^{0i} - C_{jk}^j \tilde{F}^{ki} - \frac{1}{2} C_{jk}^i \tilde{F}^{jk} + 4\pi e \tilde{u}^i. \quad (43)$$

Now concerning (41), we have, using (19) and (20):

$\tilde{\nabla}_\alpha \tilde{F}^{\alpha 0} = C_{ik}^i \tilde{F}^{k0}$, so that (41) gives the constraints equation:

$$C_{ik}^i \tilde{F}^{0k} + 4\pi e \tilde{u}^0 = 0. \quad (44)$$

- Next, observe that the Maxwell equations (4) split into the two sets:

$$\tilde{\nabla}_0 \tilde{F}_{ij} + \tilde{\nabla}_i \tilde{F}_{j0} + \tilde{\nabla}_j \tilde{F}_{0i} = 0; \quad \tilde{\nabla}_i \tilde{F}_{jk} + \tilde{\nabla}_j \tilde{F}_{ki} + \tilde{\nabla}_k \tilde{F}_{ij} = 0 \quad (45)$$

the first equation (45) gives the equation in \tilde{F}_{ij} :

$$\dot{\tilde{F}}_{ij} = -C_{ij}^k \tilde{F}_{0k} \quad (46)$$

and the second equation (45) gives the constraint equation:

$$C_{ij}^l \tilde{F}_{kl} + C_{ki}^l \tilde{F}_{jl} + C_{jk}^l \tilde{F}_{il} := C_{[ij}^l \tilde{F}_{k]l} = 0 \quad (47)$$

- Since all the indices are now fixed, we set from now, to simplify notations:

$$\begin{cases} \tilde{F}^{0i} = E^i; & \tilde{F}_{ij} = F_{ij}; & \tilde{T}_{00} = T_{00}; & \tilde{T}_{0i} = T_{0i}; & \tilde{T}_{ij} = T_{ij} \\ \tilde{\tau}_{00} = \tau_{00}; & \tilde{\tau}_{0i} = \tau_{0i}; & \tilde{\tau}_{ij} = \tau_{ij}; & \tilde{u}^0 = u^0; & \tilde{u}^i = u^i \end{cases} \quad (48)$$

- Finally, since the system in (u^i) is given by (8) for $\beta = i$, it remains to explicit the Einstein equations in (g_{ij}, k_{ij}) ; the uncharged case $F = 0$ is given by classical equations. The equations in the charged case $F \neq 0$ were set up by [16], and we easily adapt to the present case. We are then led to the following *evolution system* in $(g_{ij}, k_{ij}, E^i, F_{ij}, u^i, \Phi, U, \rho, e)$:

$$\begin{cases} \dot{g}_{ij} = -2k_{ij} \end{cases} \quad (49)$$

$$\begin{cases} \dot{k}_{ij} = R_{ij} + Hk_{ij} - 2k_j^l k_{il} - 8\pi(T_{ij} + \tau_{ij} + \rho u_i u_j) \\ + 4\pi[-T_{00} - \rho u_0^2 + g^{lm}(T_{lm} + \rho u_l u_m)]g_{ij} - \Lambda g_{ij} \end{cases} \quad (50)$$

$$\begin{cases} \dot{E}^i = HE^i - \left(C_{jk}^j g^{kl} g^{im} + \frac{1}{2}C_{jk}^i g^{jl} g^{km}\right)F_{lm} + 4\pi e u^i \end{cases} \quad (51)$$

$$\begin{cases} \dot{F}_{ij} = C_{ij}^k g_{kl} E^l \end{cases} \quad (52)$$

$$\begin{cases} \dot{u}^i = 2k_j^i u^j - \gamma_{jk}^i \frac{u^j u^k}{u^0} - 4\pi \frac{e}{\rho} E^i + 4\pi \frac{e}{\rho} \frac{g^{il} F_{jl} u^j}{u^0} \end{cases} \quad (53)$$

$$\begin{cases} \dot{\Phi} = \sqrt{2U} \end{cases} \quad (54)$$

$$\begin{cases} \dot{U} = 2HU - m^2 \Phi \sqrt{2U} \end{cases} \quad (55)$$

$$\begin{cases} \dot{\rho} = -\left[k_{ij} \frac{u^i u^j}{(u^0)^2} + C_{ij}^i \frac{u^j}{u^0} - k_i^i\right]\rho + 4\pi e g_{ij} \frac{E^j u^i}{(u^0)^2} \end{cases} \quad (56)$$

$$\begin{cases} \dot{e} = -\left[k_{ij} \frac{u^i u^j}{(u^0)^2} + C_{ij}^i \frac{u^j}{u^0} - k_i^i\right]e + 4\pi g_{ij} \frac{E^j u^i}{(u^0)^2} \frac{e^2}{\rho} \end{cases} \quad (57)$$

where:

- In (50) R_{ij} is the Ricci tensor associated to g_{ij} and given see [2] by:

$$R_{ij} = \gamma_{lm}^l \gamma_{ji}^m - \gamma_{jl}^m \gamma_{mi}^l - C_{mj}^l \gamma_{li}^m; \quad (58)$$

- In (53) γ_{jk}^i is given by (19);
- The equation (55) in U is given by (34) using the change of variable (35) which provides in the same time equation (54) in Φ following the choice $\Phi \geq 0$;
- Equation (53) in u^i is given by (8) for $\beta = i$;
- Equations (56) and (57) in ρ and e are given respectively by (30) and (37) in which \dot{u}^0 is provided by (8) for $\beta = 0$, which gives:

$$\dot{u}^0 = k_{ij} \frac{u^i u^j}{u^0} - 4\pi \frac{e}{\rho} g_{ij} \frac{u^i E^j}{u^0}.$$

Finally, (49) is provided by (9) and (50) is a direct adaptation of (14) in [16] to the present case.

- Next we have the following set of constraints equations:

$$\begin{cases} R - k_{ij}k^{ij} + H^2 = 16\pi(T_{00} + \tau_{00} + \rho u_0^2) + 2\Lambda & (59) \end{cases}$$

$$\begin{cases} \nabla^i k_{ij} = -8\pi(T_{0j} + \tau_{0j} + \rho u_0 u_j) & (60) \end{cases}$$

$$\begin{cases} C_{[ij}^l F_{k]l} := C_{ij}^l F_{kl} + C_{ki}^l F_{jl} + C_{jk}^l F_{il} = 0 & (61) \end{cases}$$

$$\begin{cases} C_{ik}^i E^k + 4\pi e u^0 = 0 & (62) \end{cases}$$

where (61) and (62) are given by (47) and (44) whereas (59) (called the Hamiltonian constraint) in which $R = g^{ij}R_{ij}$ and (60) are easily deduced from (1.20) and (1.21) in [16].

2 Study of constraints and mean curvature: the Cauchy problem

We first set up the evolution of the different quantities involved. We prove:

Lemma 2.1.

If the evolution system is satisfied, then we have:

$$\left\{ \begin{aligned} \frac{dH}{dt} &= R + H^2 + 4\pi g^{ij}(T_{ij} + \rho u_i u_j) - 12\pi(T_{00} + \rho u_0^2) - 8\pi\tau_{00} - 3\Lambda. & (63) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{dH^2}{dt} &= 2H \left[R + H^2 - 3\Lambda + 4\pi g^{ij}(T_{ij} + \rho u_i u_j) - 12\pi(T_{00} + \rho u_0^2) - 8\pi\tau_{00} \right]. & (64) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{d}{dt}(k^{ij}k_{ij}) &= 2k^{ij}R_{ij} + 2H(k^{ij}k_{ij} - \Lambda) - 16\pi k^{ij}(T_{ij} + \tau_{ij} + \rho u_i u_j) \\ &\quad + 8\pi H[-T_{00} - \rho u_0^2 + g^{lm}(T_{lm} + \rho u_l u_m)]. & (65) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{d}{dt}(T_{00} + \tau_{00} + \rho u_0^2) &= H(T_{00} + \tau_{00} + \rho u_0^2) + k^{ij}(T_{ij} + \tau_{ij} + \rho u_i u_j) \\ &\quad - \nabla_l(T^{0l} + \tau^{0l} + \rho u^0 u^l). & (66) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{d}{dt}(T^{0j} + \tau^{0j} + \rho u^0 u^j) &= H(T^{0j} + \tau^{0j} + \rho u^0 u^j) + 2k_i^j(T^{0i} + \tau^{0i} + \rho u^i u^0) \\ &\quad - \nabla_i(T^{ij} + \tau^{ij} + \rho u^i u^j). & (67) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{d}{dt}R &= 2k^{ij}R_{ij} - 2\nabla_l \nabla_i k^{il}. & (68) \end{aligned} \right.$$

Proof.

See Appendix A₂. □

Lemma 2.2.

Set:

$$\begin{cases} A = R - k_{ij}k^{ij} + H^2 - 16\pi(T_{00} + \tau_{00} + \rho u_0^2) - 2\Lambda; \\ A_j = \nabla^i k_{ij} + 8\pi(T_{0j} + \tau_{0j} + \rho u_0 u_j); \\ A_{ijk} = C_{ij}^l F_{kl} + C_{ki}^l F_{jl} + C_{jk}^l F_{il}; \\ B = C_{ik}^i E^k + 4\pi e u^0 \end{cases}$$

then

$$\left\{ \begin{array}{l} \frac{dA}{dt} = 2HA + 2g^{jm}\gamma_{mj}^k A_k \end{array} \right. \quad (69)$$

$$\left\{ \begin{array}{l} \frac{dA_j}{dt} = HA_j \end{array} \right. \quad (70)$$

$$\left\{ \begin{array}{l} \frac{dA_{ijk}}{dt} = 0 \end{array} \right. \quad (71)$$

$$\left\{ \begin{array}{l} \frac{dB}{dt} = HB. \end{array} \right. \quad (72)$$

Proof.

- 1.) (69) is a consequence of (64), (65), (66) and (68);
- 2.) (71) is a consequence of the evolution equation (52) in F_{ij} and the Jacobi polynomial relation $C_{[ij}^l C_{k]}^m = 0$;
- 3.) To obtain (72), first set:

$$C^\beta = \widetilde{\nabla}_\alpha \widetilde{F}^{\alpha\beta} - 4\pi e \widetilde{u}^\beta \quad (73)$$

for $\beta = i$, given the evolution equation (40) in $\widetilde{F}^{0i} = E^i$ which is also written in the form (43) or (51), we have

$$C^i = 0. \quad (74)$$

Next, due to the identity $\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \widetilde{F}^{\alpha\beta} = 0$ and (37) i.e $\widetilde{\nabla}_\beta (e \widetilde{u}^\beta) = 0$, we have

$$\widetilde{\nabla}_\beta C^\beta = 0; \quad (75)$$

then by a direct calculation using (19), (21) and (74), (75) gives

$$\partial_t C^0 - HC^0 = 0 \quad (76)$$

But (73) gives:

$$C^0 = \widetilde{\nabla}_\alpha \widetilde{F}^{\alpha 0} - 4\pi e \widetilde{u}^0 = -C_{ik}^i \widetilde{F}^{0k} - 4\pi e \widetilde{u}^0 = -C_{ik}^i E^k - 4\pi e \widetilde{u}^0 = -B.$$

So (72) follows from (76).

4.) For the proof of (70) see Appendix A₃.

□

Proposition 2.3.

The constraints (59), (60), (61) and (62) are satisfied in the whole domain of existence of the solutions of the evolution system (49), (50), (51), (52), (53), (54), (55), (56), (57), if and only if, they are satisfied for $t = 0$.

Proof.

Integration of (69), (70), (72) over $[0, t]$, $t > 0$ gives:

$$A_j = A_j(0) \exp\left(\int_0^t H ds\right); \quad A_{ijk} = A_{ijk}(0); \quad B = B(0) \exp\left(\int_0^t H ds\right);$$

hence $(A_j(0) = A_{ijk}(0) = B(0) = 0) \iff (A_j = A_{ijk} = B = 0)$.

Now setting $A_k = 0$ in (69) gives $\dot{A} = 2HA$ which integrates over $[0, t]$, $t > 0$ to give $A = A(0) \exp(\int_0^t 2H ds)$. Hence $(A(0) = 0) \iff (A = 0)$ and proposition 2.3 follows. □

We suppose from now on that the constraints (59), (60), (61) and (62) are satisfied for $t = 0$. As consequence, we can use the constraints which can now be considered as *properties* of the solution of the evolution system.

We now prove an important theorem on the mean curvature of the solutions of the evolution system.

Theorem 2.4.

Let $\Phi(0) > 0$, $\Lambda > -4\pi m^2(\Phi(0))^2$ be given, and suppose $H(0) = (g^{ij}k_{ij})(0) < 0$ then H is uniformly bounded and we have:

$$H(0) \leq H \leq -\sqrt{3\Lambda + 12\pi m^2(\Phi(0))^2}. \quad (77)$$

Proof.

Denote by σ_{ij} the traceless tensor associated to k_{ij} , i.e

$$k_{ij} = \frac{H}{3}g_{ij} + \sigma_{ij}. \quad (78)$$

A direct calculation gives:

$$k^{ij}k_{ij} = \frac{1}{3}H^2 + \sigma_{ij}\sigma^{ij}. \quad (79)$$

We now use the evolution equation (63) in H , in which we use the Hamiltonian constraint (59) to express the quantity $R + H^2$, (23) and (35) to express T_{00} , T_{ij} in terms of Φ and U , and finally (79) to express $k_{ij}k^{ij}$, to obtain:

$$\frac{dH}{dt} = \frac{H^2}{3} - \Lambda - 4\pi m^2\Phi^2 + \sigma_{ij}\sigma^{ij} + 16\pi U + 4\pi g^{ij}u_i u_j + 4\pi \rho u_0^2 + 8\pi \tau_{00}. \quad (80)$$

But since $\sigma_{ij}\sigma^{ij} \geq 0$, $g^{ij}u_iu_j \geq 0$, $\tau_{00} \geq 0$ (see (25)), (80) gives:

$$\frac{dH}{dt} \geq \frac{H^2}{3} - \Lambda - 4\pi m^2 \Phi^2 \quad (81)$$

Consider once more the Hamiltonian constraint (59) which gives, using (79) to express $k_{ij}k^{ij}$ and (23) to express T_{00} :

$$\frac{2}{3}H^2 - 2\Lambda - 8\pi m^2 \Phi^2 = 16\pi U + 16\pi(\tau_{00} + \rho u_0^2) + \sigma_{ij}\sigma^{ij} - R. \quad (82)$$

But it proved in [10], [26], that for the models under consideration, we always have: $R \leq 0$; (82) then gives:

$$\frac{H^2}{3} - \Lambda - 4\pi m^2 \Phi^2 \geq 0; \quad (83)$$

(81) then implies:

$$\frac{dH}{dt} \geq 0 \quad (84)$$

so that H is non-decreasing. We also deduce from (83) since by (54) we have $\dot{\Phi} \geq 0$, then $\Phi \geq \Phi(0) > 0$, that:

$$H^2 \geq 3\Lambda + 12\pi m^2 (\Phi(0))^2. \quad (85)$$

But by hypothesis, the r.h.s of (85) is strictly positive. So, since H is continuous, (85) implies:

$$H \leq -\sqrt{3\Lambda + 12\pi m^2 (\Phi(0))^2} \text{ or } H \geq \sqrt{3\Lambda + 12\pi m^2 (\Phi(0))^2}. \quad (86)$$

Also by hypothesis, $H(0) < 0$, then only the first inequality in (86) holds; moreover, (84) implies $H \geq H(0)$ and (77) follows. \square

We now introduce the Cauchy or initial value problem, taking into account (33), (35), (39); let the following quantities called *initial data* be given:

$$\begin{cases} g^0 = (g_{ij}^0) \text{ a positive definite } 3 \times 3 \text{ constant matrix;} \\ k^0 = (k_{ij}^0) \text{ a symmetric } 3 \times 3 \text{ constant matrix;} \\ F^0 = (F_{ij}^0) \text{ an antisymmetric } 3 \times 3 \text{ constant matrix;} \\ u^0 = (u^{(0),i}); \quad E^0 = (E^{0,i}), \text{ constant vectors;} \\ \Phi^0 > 0; U^0 > 0; \quad \rho^0 > 0; \quad e^0 \geq 0, \text{ real numbers;} \end{cases}$$

we look for $g = (g_{ij})$, $k = (k_{ij})$, $E = (E^i)$, $F = (F_{ij})$, $u = (u^i)$, Φ , U , ρ , e solutions of the evolution system such that:

$$\begin{cases} g(0) = g^0; \quad k(0) = k^0; \quad E(0) = E^0; \quad F(0) = F^0; \quad u(0) = u^{(0)} \\ \Phi(0) = \Phi^0; \quad U(0) = U^0; \quad \rho(0) = \rho^0; \quad e(0) = e^0. \end{cases} \quad (87)$$

By Proposition 2.3, the constraint equations (59), (60), (61), (62) are satisfied if and only if the initial data satisfy these constraints we call *initial constraints*. In what follows, we consider that it is the case.

We end this section by the useful notion of *relative norm*. Define the norm of a $n \times n$ matrix A by:

$$\|A\| = \sup \left\{ \frac{\|Ax\|}{\|x\|}, x \in \mathbb{R}^n, x \neq 0 \right\}.$$

If A_1 and A_2 are two symmetric matrices with A_1 positive definite, define the norm of A_2 with respect to A_1 by:

$$\|A_2\|_{A_1} = \sup \left\{ \frac{\|A_2 x\|}{\|A_1 x\|}, x \in \mathbb{R}^n, x \neq 0 \right\}.$$

We have the following results proved in [17]:

Lemma 2.5.

$$\|A_2\| \leq \|A_2\|_{A_1} \|A_1\|. \quad (88)$$

$$\|A_2\|_{A_1} \leq [tr(A_1^{-1} A_2 A_1^{-1} A_2)]^{\frac{1}{2}}. \quad (89)$$

Now by setting $A_1 = (g^{ij})$, $A_2 = (a_{ij})$ a direct calculation gives:

$$tr(A_1^{-1} A_2 A_1^{-1} A_2) = a^{ij} a_{ij}. \quad (90)$$

We then deduce at once from (88), (89), (90) that in these case:

$$\|A_2\| \leq \|A_1\| (a^{ij} a_{ij})^{\frac{1}{2}}. \quad (91)$$

Next, let $A = (a_{ij})$ be a $n \times n$ matrix; set $|A| = \sup\{|a_{ij}|, i, j = 1, \dots, n\}$. Then we have the following result, from [16]:

Lemma 2.6.

Let $(u^\alpha) = (u^0, u^i)$ where u^0 and u^i are link by (11). Then there exists a constant $C > 0$ such that:

$$\left| \frac{u^i}{u^0} \right| \leq C |g|^{\frac{3}{2}}; \quad |F^{0i}| \leq (g_{lm} F^{0l} F^{0m})^{\frac{1}{2}} |g|^{\frac{3}{2}}. \quad (92)$$

3 Local and global Existence of solutions

We use an iterative scheme.

3.1 Construction of the iterated sequence

We adopt the notations introduced in paragraph 2. We construct the sequence $v_n = (g_n, k_n, E_n, F_n, u_n, \Phi_n, U_n, \rho_n, e_n)$, $n \in \mathbb{N}$ as follows:

- Set $g_0 = g^0$; $k_0 = k^0$; $E_0 = E^0$; $F_0 = F^0$; $u_0 = u^{(0)}$; $\Phi_0 = \Phi^0$; $U_0 = U^0$; $\rho_0 = \rho^0$; $e_0 = e^0$.
- If $g_n, k_n, E_n, F_n, u_n, \Phi_n, U_n, \rho_n, e_n$, are known: define $\tilde{T}_{n,\alpha\beta}, \tilde{\tau}_{n,\alpha\beta}$ by substituting $\tilde{g}, \tilde{F}^{0i}, \tilde{F}_{ij}, \Phi$ in the expressions (5), (6) of $\tilde{T}_{\alpha\beta}, \tilde{\tau}_{\alpha\beta}$ by g_n, E_n, F_n, Φ_n .
- Define $v_{n+1} = (g_{n+1}, k_{n+1}, E_{n+1}, F_{n+1}, u_{n+1}, \Phi_{n+1}, U_{n+1}, \rho_{n+1}, e_{n+1})$ as solution of the *linear* ordinary differential equations (o.d.e) obtained by substituting $g, k, E, F, u, \Phi, U, \rho, e, T_{\alpha\beta}, \tau_{\alpha\beta}$ in the r.h.s of the evolution system (49) to (57), by $g_n, k_n, E_n, F_n, u_n, \phi_n, U_n, \rho_n, e_n, T_{n,\alpha\beta}, \tau_{n,\alpha\beta}$.

It is very important to notice that, for every n the initial data for the linear o.d.e's are the *same initial data* $g^0, k^0, E^0, F^0, u^{(0)}, \Phi^0, U^0, \rho^0$ and e^0 . We obtain this way a sequence $v_n = (g_n, k_n, E_n, F_n, u_n, \Phi_n, U_n, \rho_n, e_n)$ defined in a maximal interval $[0, T_n[, T_n > 0$.

3.2 Boundedness of the iterated sequence

Proposition 3.1.

*There exists $T > 0$ **independent of** n , such that the iterated sequence $v_n = (g_n, k_n, E_n, F_n, u_n, \Phi_n, U_n, \rho_n, e_n)$ is defined and uniformly bounded over $[0, T[$.*

Proof.

Let $N \in \mathbb{N}$, $N > 1$, be an integer. Suppose that we have, for $n \leq N - 1$, the inequalities:

$$\begin{cases} |g_n - g_0| \leq A_1 ; (\det g_n)^{-1} \leq A_2 ; |k_n - k_0| \leq A_3 ; |E_n - E_0| \leq A_4 \\ |F_n - F_0| \leq A_5 ; |u_n - u_0| \leq A_6 ; |\phi_n - \phi_0| \leq A_7 \\ |U_n - U_0| \leq A_8 ; |\rho_n - \rho_0| \leq A_9 ; |e_{(n)} - e_{(0)}| \leq A_{10} \end{cases} \quad (93)$$

where $A_i > 0$, $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ are given constants.

We are going to prove that one can choose the constants A_i such that (93) still holds for $n = N$ on $[0, T[, T > 0$, sufficiently small. Notice that the expression of $(g_n^{ij}) = (g_{n,ij})^{-1}$ contains $(\det g_n)^{-1}$.

- Integrating over $[0, t]$, $t > 0$, the linear o.d.e satisfied by: g_N , k_N , E_N , F_N , u_N , Φ_N , U_N , ρ_N yields:

$$\begin{cases} |g_N - g_0| \leq B_1 t; |k_N - k_0| \leq B_3 t; |E_N - E_0| \leq B_4 t; |F_N - F_0| \leq B_5 t \\ |\phi_N - \phi_0| \leq B_7 t; |U_N - U_0| \leq B_8 t; |\rho_N - \rho_0| \leq B_9 t \end{cases} \quad (94)$$

where $B_i > 0$, $i = 1, 3, 4, 5, 7, 8, 9$ are constants depending only on the A_i .

We now study the cases of $(\det g_N)^{-1}$, u_N and e_N .

- The iterated equation satisfied by g_N writes, using (49):

$$\dot{g}_{N,ij} = -2k_{N-1,ij}. \quad (95)$$

Recall the formula:

$$\frac{d}{dt}[\ln(\det g_N)] = g_N^{ij} \partial_t g_{N,ij}; \quad (96)$$

on the other hand we have:

$$\frac{d}{dt}[\ln(\det g_N)] = (\det g_N)^{-1} \frac{d}{dt}(\det g_N) = -(\det g_N) \frac{d}{dt}(\det g_N)^{-1}. \quad (97)$$

(97) and (96) then give, using (95):

$$\frac{d}{dt}(\det g_N)^{-1} = (2g_N^{ij} k_{N-1,ij})(\det g_N)^{-1}$$

an o.d.e in $(\det g_N)^{-1}$ which integrate at once over $[0, t]$, $t > 0$ to give:

$$(\det g_N)^{-1} = (\det g^0)^{-1} \exp\left(\int_0^t 2(g_N^{ij} k_{N-1,ij})(s) ds\right) \quad (98)$$

Now (95) which is analogous to (49) shows that g_N and k_{N-1} are the first and second fundamental forms of a space-like hypersurface; so $g_N^{ij} k_{N-1,ij} = k_{N-1,i}^i = \text{tr}(k_{N-1})$. We then deduce from (98) using (94), that:

$$(\det g_N)^{-1} \leq (\det g^0)^{-1} \exp(C_1 t) \quad (99)$$

where $C_1 > 0$ is a constant depending only on A_i and $|g^0|$, $|k^0|$. Hence, using (99), it appears that if we take in (93): $A_2 > (\det g^0)^{-1}$ i.e $(\det g^0)A_2 > 1$, then given the continuity of $t \mapsto \exp(C_1 t)$, we will have for t sufficiently small $(\det g^0)A_2 > \exp(C_1 t) > 1$. Then, there exists $t_1 > 0$ such that, for $0 < t < t_1$ we have, using (99):

$$(\det g_N)^{-1} \leq A_2. \quad (100)$$

- Next, in order to have for u_N and e_N inequalities analogous to (94), we need to bound $\frac{1}{\rho_n}$ which appears in the iterated equations in u_{n+1} , e_{n+1} , built from equations (53) and (57) in u and e . Now following the definition of the iterated sequence (v_n) and given equation (56) in ρ , ρ_{n+1} satisfies, for $0 \leq n \leq N-1$

$$\dot{\rho}_{n+1} = G_n \quad (101)$$

where

$$G_n = - \left[k_{n,ij} \frac{u_n^i u_n^j}{(u_n^0)^2} + C_{ij}^i \frac{u_n^j}{u_n^0} - k_{n,i}^i \right] \rho_n + 4\pi e_n g_{n,ij} \frac{u_n^i E_n^j}{(u_n^0)^2}. \quad (102)$$

We have in the expression (102) of G_n , using (92):

$$\left| \frac{u_n^i}{u_n^0} \right| \leq C |g_n|^{\frac{3}{2}}, \quad \text{and} \quad \frac{|u_n^i|}{(u_n^0)^2} = \frac{|u_n^i|}{u_n^0} \frac{1}{u_n^0} \leq \frac{|u_n^i|}{u_n^0} \quad \text{since } u_n^0 \geq 1$$

Then, by (93), G_n is bounded, i.e, $\exists C > 0$, $|G_n| < C$; hence, $|\dot{\rho}_{n+1}| \leq C$ and this implies:

$$\dot{\rho}_{n+1} \geq -C. \quad (103)$$

Integrating (103) over $[0, t]$, $t > 0$ yields $\rho_{n+1} \geq \rho^0 - Ct$ but since $\rho^0 > 0$, there exists $t_2 > 0$ such that for $t \in [0, t_2]$ we have $Ct < \frac{\rho^0}{2}$; then

$$(0 \leq t \leq t_2) \implies (\rho_{n+1} \geq \frac{\rho^0}{2})$$

thus, $\frac{1}{\rho_{n+1}} \leq \frac{2}{\rho^0}$ and the sequence

$$\left(\frac{1}{\rho_n} \right)$$

is bounded. Hence u_N and e_N also satisfy:

$$|u_N - u_0| \leq B_6 t; \quad |e_N - e_0| \leq B_{10} t \quad (104)$$

where as in (104) B_6 , B_{10} are constant depending only on the A_i . We then conclude that if $T > 0$ is such that:

$$T < \inf(t_1, t_2), \quad B_i T < A_i, \quad i = 1, 2, \dots, 10,$$

then, by (94) and (104) v_N also satisfies (93) on $[0, T[$. Hence, the iterated sequence (v_n) is uniformly bounded over $[0, T[$. \square

3.3 Local existence of solutions

Theorem 3.2.

The initial value problem for the Einstein-Maxwell-Scalar Field system has a unique local solution.

Proof.

We prove that the iterated sequence (v_n) converges uniformly on each bounded interval $[0, \delta] \subset [0, T]$, $\delta > 0$. For this purpose, we study the difference $v_{n+1} - v_n$. But given the evolution equations (54) and (55) in Φ and U , we will deal with the difference:

$$\sqrt{2U_{n+1}} - \sqrt{U_n} = \frac{2(U_{n+1} - U_n)}{\sqrt{2U_{n+1}} + \sqrt{U_n}}.$$

We then need to show first of all that the sequence

$$\left(\frac{1}{\sqrt{2U_n}} \right)$$

is uniformly bounded.

- By (55), the iterated equation providing U_{n+1} writes:

$$\dot{U}_{n+1} = 2H_n U_n - m^2 \Phi_n \sqrt{2U_n}. \quad (105)$$

But by Proposition 3.1, there exists a constant $C > 0$ such that we have over $[0, T[$:

$$|2H_n U_n - m^2 \Phi_n \sqrt{2U_n}| \leq C$$

(105) then gives:

$$\frac{dU_{n+1}}{dt} \geq -C$$

and integrating over $[0, t]$, $0 \leq t < T$ yields:

$$U_{n+1} \geq U^0 - Ct.$$

Recall that $U^0 > 0$. Then taking t sufficiently small such that $Ct \leq \frac{U^0}{2}$ we have $U_{n+1} \geq \frac{U^0}{2}$. Then

$$\frac{1}{\sqrt{2U_{n+1}}} \leq \frac{1}{\sqrt{U^0}}$$

which shows that $\left(\frac{1}{\sqrt{2U_n}} \right)$ is uniformly bounded over $[0, T[$, $T > 0$ small enough.

- Since $(\frac{1}{\rho_n})$ is also bounded, taking the difference between two consecutive iterated equations, we deduce from the evolution system, using $v_n(0) = v^0, \forall n$, that there exists a constant $C_2 > 0$ such that:

$$\begin{aligned}
& |g_{n+1} - g_n| + |k_{n+1} - k_n| + |E_{n+1} - E_n| + |F_{n+1} - F_n| + |u_{n+1} - u_n| + \\
& |\Phi_{n+1} - \Phi_n| + |U_{n+1} - U_n| + |\rho_{n+1} - \rho_n| + |e_{n+1} - e_n| \leq \\
& C_2 \int_0^t \left(|g_n - g_{n-1}| + |k_n - k_{n-1}| + |E_n - E_{n-1}| + |F_n - F_{n-1}| + \right. \\
& |u_n - u_{n-1}| + |\Phi_n - \Phi_{n-1}| + |U_n - U_{n-1}| + |\rho_n - \rho_{n-1}| + \\
& \left. |e_n - e_{n-1}| \right) (s) ds.
\end{aligned} \tag{106}$$

So, if we set:

$$\begin{aligned}
\alpha_n = & |g_{n+1} - g_n| + |k_{n+1} - k_n| + |F_{n+1} - F_n| + |E_{n+1} - E_n| + |u_{n+1} - u_n| + \\
& |\Phi_{n+1} - \Phi_n| + |U_{n+1} - U_n| + |\rho_{n+1} - \rho_n| + |e_{n+1} - e_n|
\end{aligned} \tag{107}$$

then (106) shows that we have:

$$\alpha_n(t) \leq C_2 \int_0^t \alpha_{n-1}(s) ds. \tag{108}$$

(108) gives, by an immediate induction on n :

$$\alpha_n(t) \leq \|\alpha_2\|_\infty \frac{(C_2 t)^{n-2}}{(n-2)!}. \tag{109}$$

But since the series $\sum_{n=0}^{+\infty} \frac{C_2^n}{n!}$ is convergent we have necessarily $\alpha_n(t) \rightarrow 0$ as $n \rightarrow +\infty$. Definition (107) of α_n then shows that each of the sequences (g_n) , (k_n) , (E_n) , (F_n) , (u_n) , (Φ_n) , (U_n) , (ρ_n) , (e_n) converges uniformly on each bounded interval $[0, \delta]$, $0 < \delta < T$ and that their respective limits denoted: g , k , E , F , u , Φ , U , ρ and e are continuous function of t . From the iterated equations, it appears immediately that there exists a constant $C_3 > 0$ such that:

$$\left| \frac{dv_{n+1}}{dt} - \frac{dv_n}{dt} \right| \leq C_3 |v_{n+1} - v_n| \tag{110}$$

where we set, for $v = (v^i)$, $|v| = \sum |v^i|$. The convergence of (v_n) then implies, given (110), the convergence of

$$\left(\frac{dv_n}{dt} \right)$$

and hence, that each of the sequences (\dot{g}_n) , (\dot{k}_n) , (\dot{E}_n) , (\dot{F}_n) , (\dot{u}_n) , $(\dot{\Phi}_n)$, (\dot{U}_n) , $(\dot{\rho}_n)$, (\dot{e}_n) converges uniformly on each interval $[0, \delta]$, $0 < \delta < T$.

Consequently the limit functions $g, k, E, F, u, \Phi, U, \rho$ and e are of class \mathcal{C}^1 and $v := (g, k, E, F, u, \Phi, U, \rho, e)$ is a local solution of the coupled Einstein-Maxwell-Massive Scalar Field system.

Now to prove the *uniqueness* of the solution, suppose v_1 and v_2 are two solutions of the Cauchy problem, with the *same initial data*. Then, defining $\alpha(t)$ the same way as α_n (see (107)) for the difference $|v_1 - v_2|$, leads, using the evolution system to:

$$\alpha(t) \leq \int_0^t \alpha(s) ds$$

which gives by Gronwall Lemma $\alpha = 0$; hence $v_1 = v_2$ and uniqueness follows. □

3.4 Global existence of solutions

We prove:

Theorem 3.3.

Let $\Lambda > -4\pi m^2(\Phi^0)^2$ be given and suppose $H^0 := (g^0)^{ij}k_{ij}^0 < 0$. Then the initial value problem for the Einstein-Maxwell-Massive Scalar Field system has a unique global solution defined all over the interval $[0, +\infty[$.

Proof.

Following the standard theory on the first order differential systems, it will be enough if we could prove, given the evolution system (49) to (57) that, if each of the functions: $|g|, |k|, |E|, |F|, |\Phi|, |u^i|, |\rho|, |e|, |R_{ij}|, |R|, (detg)^{-1}, |\frac{1}{\rho}|$, is uniformly bounded over every bounded interval $[0, T^*[$, where $T^* < +\infty$.

Notice that the hypothesis of Theorem 2.4 are satisfied; so (77) applies, i.e H is bounded.

- We deduce at once from (83) and using (77) that Φ is bounded.
- Since the l.h.s of (82) is bounded and since $-R > 0, \tau_{00} \geq 0, \sigma^{ij}\sigma_{ij} \geq 0, U \geq 0, \rho \geq 0$; we deduce that U and $\rho u_0^2 = \rho(u_0)^2$ are bounded; but $\rho(u_0)^2 \geq \rho \geq 0$, since $u^0 \geq 1$; then we deduce that ρ is bounded.
- (80) gives, using (79) to express $\sigma^{ij}\sigma_{ij}$, and since: $U \geq 0, g^{ij}u_i u_j \geq 0, \rho u_0^2 \geq 0, \tau_{00} \geq 0$:

$$\frac{dH}{dt} \geq k_{ij}k^{ij} - \Lambda - 4\pi m^2 \Phi^2. \quad (111)$$

Then integrating (111) over $[0, t]$, $0 < t \leq T^*$, we have, since H and Φ are bounded:

$$\int_0^{T^*} k_{ij}k^{ij}(s)ds < +\infty \quad (112)$$

Next we have, integrating (49) over $[0, t]$, $t \in [0, T^*]$:

$$|g(t)| \leq |g^0| + 2 \int_0^t |k(s)| ds \quad (113)$$

but setting $A_1 = (g^{ij})$, $A_2 = (k_{ij})$, (91) gives:

$$\|k\| \leq \|g\| (k_{ij} k^{ij})^{\frac{1}{2}} \quad (114)$$

and we deduce from (113) and (114) that:

$$\|g(t)\| \leq \|g^0\| + 2 \int_0^t \|g(s)\| (k_{ij} k^{ij})^{\frac{1}{2}}(s) ds.$$

Hence, by Gronwall Lemma, there exists a constant $C > 0$ such that:

$$\|g(t)\| \leq C \|g^0\| \exp(C \int_0^t (k_{ij} k^{ij})^{\frac{1}{2}}(s) ds) \quad (115)$$

but we deduce from (115) applying Schwarz inequality, using (112) and since $T^* < +\infty$, that $\|g\|$ and hence $|g|$ is bounded.

- By (58), R_{ij} expresses in terms of γ_{ij}^k given itself by (19), which involves g^{ij} ; so we need to control $(\det g)^{-1}$. We use once more the formula:

$$\frac{d}{dt} [\ln(\det g)] = g^{ij} \frac{dg^{ij}}{dt}.$$

Then, using the evolution equation (49), we obtain:

$$\frac{d}{dt} [\ln(\det g)] = -2H. \quad (116)$$

Since H is bounded, we obtain, by integrating (116):

$$-C \leq \ln(\det g) \leq C$$

where $C > 0$ is a constant. Hence:

$$e^{-C} \leq \det g \leq e^C$$

which shows that, both $\det g$ and $(\det g)^{-1}$ are bounded. Then, by (58), R_{ij} is bounded and $R = g^{ij} R_{ij}$ is bounded.

- Now consider the Hamiltonian constraint (59) which gives, since $\tau_{00} \geq 0$, $\rho u_0^2 \geq 0$, $T_{00} \geq 0$ (see (23))

$$R + H^2 - 2\Lambda \geq k_{ij} k^{ij}.$$

Then, since R and H^2 are bounded, $k_{ij} k^{ij}$ which is positive is bounded. Then, since $\|g\|$ is bounded, by (114), $\|k\|$ and hence $|k|$, is bounded.

- Deduce from (82) whose l.h.s is bounded and using $U \geq 0$, $\tau_{00} \geq 0$, $\rho u_0^2 \geq 0$, $\sigma^{ij}\sigma_{ij} \geq 0$, $-R > 0$, that τ_{00} is bounded. But by (25):

$$\tau_{00} = \frac{1}{2}g_{ij}E^iE^j + \frac{1}{4}F^{ij}F_{ij} \quad (117)$$

which implies:

$$0 \leq \frac{1}{2}g_{ij}E^iE^j \leq \tau_{00} \quad ; \quad 0 \leq \frac{1}{4}F^{ij}F_{ij} \leq \tau_{00} \quad (118)$$

and $g_{ij}E^iE^j$ is bounded. Hence, using (92) and since $|g|$ is bounded, E^i is bounded.

- The constraint (62) shows, since $u^0 \geq 1$ and E^i is bounded, that e is bounded.
- Integrating the equation (52) in F_{ij} over $[0, t]$, $t \leq T^* < +\infty$ shows, since $|g|$ and $|E|$ are bounded, that $|F|$ is bounded.
- It remains the cases of $\frac{1}{\rho}$ and u^i .
Expression (32) of ρ gives, using the notations (48):

$$\left(\frac{1}{\rho}\right)(t) = \frac{u^0}{\rho^0 u^0(0)} \exp \left[\int_0^t (-H + C_{ij}^i \frac{u^j}{u^0})(s) ds \right]. \quad (119)$$

Now as we already indicated, equation (8) gives for $\beta = 0$

$$\dot{u}^0 = k_{ij} \frac{u^i u^j}{u^0} - \left(4\pi e g_{ij} \frac{u^i E^j}{u^0} \right) \frac{1}{\rho}. \quad (120)$$

We deduce from (119) and (120) that:

$$\dot{u}^0 \leq G(t) u^0 \quad (121)$$

where:

$$G(t) = \frac{|k_{ij} u^i u^j|}{(u^0)^2} + \frac{|4\pi e g_{ij} E^j|}{\rho^0 u^0(0)} \cdot \frac{|u^i|}{u^0} \exp \left[\int_0^t (-H + C_{ij}^i \frac{u^j}{u^0})(s) ds \right]. \quad (122)$$

Integrating (121) over $[0, t]$, $t \leq T^* < +\infty$ gives:

$$u^0(t) \leq u^0(0) \exp \left[\int_0^t G(s) ds \right] \quad (123)$$

then, using (92) to bound $\frac{u^i}{u^0}$ and since k , e , g , E , H are bounded, by (123), there exists a constant $C(T^*) > 0$ such that $u^0 \leq C(T^*) u^0(0)$. Hence by (119), $\frac{1}{\rho}$ is bounded and writing $u^i = \frac{u^i}{u^0} u^0$ shows that u^i is bounded. This completes the proof of theorem 3.3

□

4 Asymptotic behaviour

We consider the global solution over $[0, +\infty[$ and we investigate the asymptotic behaviour of the different elements at late times. We introduce the following quantity which plays a key role:

$$Q = H^2 - 24\pi T_{00} - 3\Lambda \quad (124)$$

At late times, we have the following asymptotic behaviour:

Theorem 4.1.

$$Q = \mathcal{O}(e^{-2\gamma t}) \quad (125)$$

$$\sigma_{ij}\sigma^{ij} = \mathcal{O}(e^{-2\gamma t}) \quad (126)$$

$$|R| = \mathcal{O}(e^{-2\gamma t}) \quad (127)$$

$$\tau_{00} = \mathcal{O}(e^{-2\gamma t}) \quad (128)$$

$$\rho = \mathcal{O}(e^{-2\gamma t}) \quad (129)$$

$$(\dot{\Phi})^2 = \mathcal{O}(e^{-2\gamma t}) \quad (130)$$

$$F^{ij}F_{ij} = \mathcal{O}(e^{-2\gamma t}) \quad (131)$$

$$E^i E_i = \mathcal{O}(e^{-2\gamma t}) \quad (132)$$

$$\|\sigma(t)\| \leq C e^{-\gamma t} \|g(t)\| \quad (133)$$

$$\Phi^2 \longrightarrow L > 0 \quad (134)$$

$$T_{00} \longrightarrow \frac{m^2}{2} L \quad (135)$$

$$H = -(3C_0)^{\frac{1}{2}} + \mathcal{O}(e^{-2\delta t}) \quad (136)$$

$$|e^{-2\delta t} g_{ij}| \leq C \quad (137)$$

$$|e^{2\delta t} g^{ij}| \leq C \quad (138)$$

$$g_{ij}(t) = e^{2\delta t} (G_{ij} + \mathcal{O}(e^{-\gamma t})) \quad (139)$$

$$g^{ij}(t) = e^{-2\delta t} (G^{ij} + \mathcal{O}(e^{-\gamma t})) \quad (140)$$

$$|k_{ij}| \leq C e^{2\delta t} \quad (141)$$

$$|E^i| \leq C e^{\nu t} \quad (142)$$

$$|F_{ij}| \leq C e^{(2\delta+\nu)t} \quad (143)$$

$$|F^{ij}| \leq C e^{(\delta-\gamma)t} \quad (144)$$

$$|e| \leq C e^{\nu t} \quad (145)$$

where

$$\begin{aligned} \gamma &= \left[\frac{\Lambda + 4\pi m^2 (\Phi^0)^2}{3} \right]^{\frac{1}{2}}; \quad C_0 = \Lambda + 4\pi m^2 L; \quad \delta = \left[\frac{1}{3} (\Lambda + 4\pi m^2 L) \right]^{\frac{1}{2}}; \\ \nu &= 3\delta - \gamma; \end{aligned} \quad (146)$$

G_{ij} a symmetric positive definite constant matrix; $(G^{ij}) = (G_{ij})^{-1}$.

Proof.

Notice that by (23) and (35), we have: $m^2\Phi^2 = 2T_{00} - 2U$. Expression (124) of Q then shows, using (82) that $Q \geq 0$. Let us point out first of all that, the quantity Q defined by (124) plays a key role in GR, and in the presence of the massive scalar field, it stands for the quantities S in [13], Z in [20], \tilde{S} in [11], \bar{S} in [7] and reduces to $H^2 \pm 3\Lambda$ in [26] which deals with the case of zero scalar field.

- We have, using the expression (23) of T_{00} and (35):

$$T_{00} = U + \frac{1}{2}m^2\Phi^2 \quad (147)$$

then, the evolution equations (54) and (55) in Φ and U give:

$$\dot{T}_{00} = 2HU. \quad (148)$$

Expression (124) of Q then gives, using (148):

$$\dot{Q} = 2H(\dot{H} - 24\pi U)$$

then, using equation (63) in H , in which we use the Hamiltonian constraint (59) to express $R + H^2$ and (79) to express $k_{ij}k^{ij}$ and since by (23) and (35): $4\pi g^{ij}T_{ij} = 24\pi U - 12T_{00}$, we obtain

$$\dot{Q} = \frac{2}{3}H \left[H^2 - 24\pi T_{00} - 3\Lambda + 3(8\pi\tau_{00} + 4\pi\rho u_0^2 + 4\pi g^{ij}u_i u_j + \sigma_{ij}\sigma^{ij}) \right]. \quad (149)$$

But since $8\pi\tau_{00} + 4\pi\rho u_0^2 + 4\pi g^{ij}u_i u_j + \sigma_{ij}\sigma^{ij} \geq 0$, $H < 0$, and given the definition (124) of Q , (149) gives:

$$\dot{Q} \leq \frac{2}{3}HQ. \quad (150)$$

Integrating (150) over $[0, t]$, $t > 0$ yields:

$$0 \leq Q \leq Q_0 \exp\left[\int_0^t \frac{2}{3}H ds\right];$$

and (77) gives

$$0 \leq Q \leq Q_0 \exp\left(-2t\sqrt{\frac{1}{3}(\Lambda + 4\pi m^2(\Phi^0)^2)}\right)$$

and (125) follows.

- Using (82), the results (126), (127), (128), (129) and (130) (since $(\dot{\Phi})^2 = 2U$) are direct consequences of (125).

- (131) and (132) are consequences of (128), using (118).
- Setting $A_1 = (g^{ij})$ and $A_2 = (\sigma_{ij})$, (91) gives:

$$\|\sigma(t)\| \leq \|g(t)\|(\sigma_{ij}\sigma^{ij})^{\frac{1}{2}} \quad (151)$$

(133) then follows from (126).

- The evolution equations (54) and (55) in Φ and U give:

$$m^2\Phi\dot{\Phi} + \dot{U} = \frac{d}{dt}\left[\frac{m^2}{2}\Phi^2 + U\right] = 2HU. \quad (152)$$

But since $H < 0$ and $U > 0$, (152) implies: $\frac{d}{dt}[\frac{m^2}{2}\Phi^2 + U] \leq 0$; we then deduce, using $U > 0$ that:

$$\frac{m^2}{2}\Phi^2 \leq \frac{m^2}{2}\Phi^2 + U \leq \frac{m^2}{2}(\Phi^0)^2 + U^0. \quad (153)$$

Hence Φ^2 is bounded. But the evolution equation (54) in Φ shows that $\dot{\Phi} > 0$; then $\Phi \geq \Phi^0 > 0$ and since $\frac{d}{dt}(\Phi^2) = 2\Phi\dot{\Phi} > 0$, Φ^2 is an increasing function. Φ^2 being positive, increasing and bounded has a strictly positive limit, i.e, there exists $L > 0$ such that

$$\Phi^2 \longrightarrow L \quad (154)$$

with:

$$\Phi^2 \leq L \quad (155)$$

and we have (134).

- (135) follows from (147), (130) and (134).
- To prove (136) which is one for the main results, since by (125) $Q \longrightarrow 0$ its expression (124) shows, using (135) that:

$$H^2 - 3\Lambda \longrightarrow 12\pi m^2 L. \quad (156)$$

Hence:

$$H^2 - (3\Lambda + 12\pi m^2 L) = \left[H - (3\Lambda + 12\pi m^2 L)^{\frac{1}{2}}\right] \left[H + (3\Lambda + 12\pi m^2 L)^{\frac{1}{2}}\right] \longrightarrow 0.$$

But by (77), $H < 0$; so:

$$H - (3\Lambda + 12\pi m^2 L)^{\frac{1}{2}} < -(3\Lambda + 12\pi m^2 L)^{\frac{1}{2}} < 0.$$

We then deduce that:

$$H \longrightarrow -(3C_0)^{\frac{1}{2}} \quad (157)$$

where:

$$C_0 = \Lambda + 4\pi m^2 L > 0. \quad (158)$$

But since by (84) H is an increasing function (157) implies:

$$H \leq -(3C_0)^{\frac{1}{2}}. \quad (159)$$

Now (155): $-4\pi m^2 \Phi^2 \geq -4\pi m^2 L$; (81) then implies:

$$\frac{dH}{dt} \geq \frac{H^2}{3} - C_0 \quad (160)$$

write:

$$\frac{H^2}{3} - C_0 = \frac{1}{3}(H^2 - 3C_0) = \frac{1}{3}[-H + (3C_0)^{\frac{1}{2}}][-H - (3C_0)^{\frac{1}{2}}]$$

in which using (159) we have:

$$-H + (3C_0)^{\frac{1}{2}} \geq (3C_0)^{\frac{1}{2}} + (3C_0)^{\frac{1}{2}} = 2(3C_0)^{\frac{1}{2}};$$

(159) also implies: $-H - (3C_0)^{\frac{1}{2}} \geq 0$. We then deduce from (160)

$$\frac{dH}{dt} \geq 2\delta[-H - (3C_0)^{\frac{1}{2}}] \quad (161)$$

where:

$$\delta = \frac{1}{3}(3C_0)^{\frac{1}{2}}. \quad (162)$$

Write (161) in the form:

$$\frac{d}{dt}(H + (3C_0)^{\frac{1}{2}}) + 2\delta[H + (3C_0)^{\frac{1}{2}}] \geq 0. \quad (163)$$

Multiply (163) by $e^{2\delta t} > 0$ and integrate over $[0, t]$ to obtain:

$$e^{2\delta t}[H + (3C_0)^{\frac{1}{2}}] \geq H(0) + (3C_0)^{\frac{1}{2}}. \quad (164)$$

Now multiply (164) by $-e^{-2\delta t} < 0$, use once more (159) which gives $H + (3C_0)^{\frac{1}{2}} < 0$ to obtain:

$$|H + (3C_0)^{\frac{1}{2}}| \leq |H(0) + (3C_0)^{\frac{1}{2}}|e^{-2\delta t}$$

and (136) follows with , see (158) and (162): $\delta = [\frac{1}{3}(\Lambda + 4\pi m^2 L)]^{\frac{1}{2}}$.

- To prove (137), set $h_{ij} = e^{-2\delta t}g_{ij}$; then we have, using equation (49) in g_{ij} :

$$\frac{dh_{ij}}{dt} = -2\delta h_{ij} - 2k_{ij}e^{-2\delta t}. \quad (165)$$

Now, using (78) to express k_{ij} , the result (136), the expression of C_0 and δ in (146) we deduce from (165) that:

$$\frac{dh_{ij}}{dt} = \mathcal{O}(e^{-2\delta t})h_{ij} - 2e^{-2\delta t}\sigma_{ij}. \quad (166)$$

Integrating (166) over $[0, t]$, $t > 0$, and taking the norm yields:

$$\|h(t)\| \leq \|h(0)\| + C \int_0^t (e^{-2\delta s}\|h(s)\| + e^{-2\delta s}\|\sigma(s)\|)ds. \quad (167)$$

where $C > 0$ is a constant. Notice that $h^{ij} = e^{2\delta t}g^{ij}$. Now setting in formula (91): $A_1 = (h^{ij})$, $A_2 = (\sigma_{ij})$ yields:

$$\|\sigma(s)\| \leq \|h(s)\|(\sigma_h^{ij}\sigma_{ij})^{\frac{1}{2}}$$

where $\sigma_h^{ij} = h^{il}h^{jk}\sigma_{lk} = e^{4\delta t}\sigma^{ij}$. Hence, $\|\sigma(s)\| \leq \|h(s)\|e^{2\delta t}(\sigma^{ij}\sigma_{ij})^{\frac{1}{2}}$; (126) then gives

$$\|\sigma(s)\| \leq \|h(s)\|e^{2\delta s}e^{-\gamma s}.$$

We then deduce from (167):

$$\|h(t)\| \leq \|h(0)\| + C \int_0^t (e^{-2\delta s}\|h(s)\| + e^{-\gamma s}\|h(s)\|)ds$$

then, since $\gamma < \delta$ [see (146)]:

$$\|h(t)\| \leq \|h(0)\| + C \int_0^t e^{-\gamma s}\|h(s)\|ds.$$

By Gronwall Lemma, this gives:

$$\|h(t)\| \leq \|h(0)\| \exp\left[\int_0^t Ce^{-\gamma s}\right] \leq C,$$

where $C > 0$ is a constant, and (137) follows.

- To obtain (138) set this time $L^{ij} = e^{2\delta t}g^{ij}$ and proceed as for (136) and obtain (138).
- To prove (139) which is one of the main results, first use

$$e^{-2\delta t}\sigma_{ij} = \mathcal{O}(e^{-\gamma t}) \quad (168)$$

which is a direct consequence of (151), (126) and (137). Recall that setting $h_{ij} = e^{-2\delta t} g_{ij}$ led to (166). We deduce from (166), using (168), $\gamma < \delta$, and since h is bounded:

$$\frac{dh_{ij}}{dt} = \mathcal{O}(e^{-\gamma t}). \quad (169)$$

(169) shows that \dot{h}_{ij} has an exponential fall of at late times and by the mean value theorem, h_{ij} has a limit we denote G_{ij} as $t \rightarrow +\infty$. Then we can write:

$$h_{ij}(t) = G_{ij} + \mathcal{O}(e^{-\gamma t}) \quad (170)$$

where, given the properties of h_{ij} , G_{ij} is a symmetric, positive definite constant 3×3 matrix. (139) follows from (170) since $g_{ij} = e^{2\delta t} h_{ij}$.

- (140) is a direct consequence of (139) since $(g^{ij}) = (g_{ij})^{-1}$.
- To prove (141), use (78) which implies, since H is bounded

$$\|k\| \leq C(\|g\| + \|\sigma\|). \quad (171)$$

(141) then follows from (171), using (139) and (168).

- To obtain (142), use (92) which gives $|E^i| \leq (E^j E_j)^{\frac{1}{2}} |g|^{\frac{3}{2}}$ and conclude by applying (132) and (139).
- To obtain (143), integrate equation (52) in F_{ij} and use (139) and (142).
- To obtain (144), use $F^{ij} = g^{il} g^{jk} F_{kl}$, (140) and (143).
- Finally to obtain (145), use the constraint equation (62), (142) and $u^0 \geq 1$. This completes the proof of Theorem 4.1.

□

Remark 4.2.

The result (139) shows an exponential growth of the metric tensor g at late times. this result, but also (136) confirm mathematically the accelerated expansion of the universe as observed in Astrophysics.

5 Geodesic Completeness

We prove:

Theorem 5.1.

The space-time which exists globally is future geodesically complete.

Proof.

We use the fact that, the geodesics equations for the metric (1) imply that, along the geodesics whose affine parameter is denoted by s , the variables t , u^0 , u^i satisfy a first order differential system containing between others, the equation:

$$\frac{dt}{ds} = u^0. \quad (172)$$

The space-time will be future geodesically complete if we prove that the affine parameter s also goes to infinity. Then using (11), the notations (48) and (172), it will enough if we could prove that:

$$\frac{ds}{dt} = (1 + g_{ij}u^i u^j)^{-\frac{1}{2}} \geq C > 0 \quad (173)$$

where $C > 0$ is a constant, since one could then deduce at once from (173), integrating, that $s \geq Ct + D$ where D is a constant; hence $s \rightarrow +\infty$ as $t \rightarrow +\infty$. Our goal will then be to prove, that $(1 + g_{ij}u^i u^j)$ or finally $g_{ij}u^i u^j$, is bounded by a strictly positive constant. For this purpose we use equation (53) in u^i . It shows to be useful considering $u_i = g_{ij}u^j$, rather than u^i . Differentiating this relation, we have:

$$\frac{du_i}{dt} = g_{ij} \frac{du^j}{dt} + u^j \frac{dg_{ij}}{dt} \quad (174)$$

then, using equation (53) in u^i and equation (49) in g_{ij} , we deduce from (174), the equation:

$$\frac{du^i}{dt} = -g_{ij}\gamma_{lk}^j \frac{u^l u^k}{u^0} + g_{ij}(-4\pi \frac{e}{\rho} E^j + 4\pi \frac{e}{\rho} F_l^j \frac{u^l}{u^0}). \quad (175)$$

In order to bound $(g_{ij}u^i u^j)$ we set up a differential equation for this quantity. First notice that equation (49) gives, using $g^{ij}g_{il} = \delta_l^j$:

$$\frac{dg^{ij}}{dt} = 2k^{ij};$$

from where we deduce:

$$\frac{d}{dt}(g^{ij}u_i u_j) = 2k^{ij}u_i u_j + 2g^{ij}u_j \frac{du_i}{dt}. \quad (176)$$

A direct calculation using (175) shows that, in (176) we have:

$$2g^{ij}u_j \frac{du_i}{dt} = -8\pi \frac{e}{\rho} u_i E^i + 8\pi \frac{e}{\rho} F_{lk} \frac{u^l u^k}{u^0} - 2\gamma_{jk}^i \frac{u^j u^k u_i}{u^0}. \quad (177)$$

But given the antisymmetry of F_{lk} , we have $F_{lk}u^l u^k = 0$. Next using the expression (19) of γ_{ij}^l we obtain for the last term in (177):

$$\gamma_{jk}^i u^j u^k u_i = -\frac{1}{2}g_{jl}C_{kp}^l u^j u^k u^p + \frac{1}{2}g_{kl}C_{pj}^l u^j u^k u^p + \frac{1}{2}C_{jk}^i u^j u^k u_i. \quad (178)$$

But since $C_{jk}^l = -C_{kj}^l$ the last term in (178) vanishes. For the same reason:

$$-\frac{1}{2}g_{jl}C_{kp}^l u^j u^k u^p + \frac{1}{2}g_{kl}C_{pj}^l u^j u^k u^p = \frac{1}{2}(C_{pk}^l + C_{kp}^l)g_{jl}u^j u^k u^p = 0.$$

Consequently, the r.h.s of (177) reduces to its first term and (176) gives:

$$\frac{d}{dt}(g^{ij}u_i u_j) = 2k^{ij}u_i u_j - 8\pi \frac{e}{\rho} u_i E^i. \quad (179)$$

But by (78), $k^{ij} = \frac{H}{3}g^{ij} + \sigma^{ij}$; (179) then gives, using (136) and (146):

$$\frac{d}{dt}(g^{ij}u_i u_j) = \left[-2\delta + \mathcal{O}(e^{-2\delta t}) \right] g^{ij}u_i u_j + 2\sigma^{ij}u_i u_j - 8\pi \frac{e}{\rho} u_i E^i. \quad (180)$$

Now we have $\sigma^{ij} = g^{ik}g^{jl}\sigma_{kl}$, so by (168) and (140): $e^{(2\delta+\gamma)t}\sigma^{ij}$ is bounded. This implies, since the matrix G^{ij} is positive definite and constant, that there exists a constant $C > 0$ such that:

$$\sigma^{ij}u_i u_j \leq C e^{-(2\delta+\gamma)t} G^{ij}u_i u_j. \quad (181)$$

Now by (140), there exists a constant $C > 0$ such that:

$$G^{ij}u_i u_j \leq C e^{2\delta t} g^{ij}u_i u_j. \quad (182)$$

Now deduce from (32) and (38) that:

$$\frac{e}{\rho} = \frac{e^0}{\rho^0}, \quad (183)$$

we then obtain from (180), using (181), (182) and (183):

$$\frac{d}{dt}(g^{ij}u_i u_j) \leq (-2\delta + C e^{-2\delta t})g^{ij}u_i u_j + C e^{-\gamma t} g^{ij}u_i u_j + C |u_i E^i|. \quad (184)$$

Now set:

$$W = e^{2\delta t} g^{ij}u_i u_j \quad (185)$$

then we have, using (184):

$$\begin{aligned} \frac{dW}{dt} &= 2\delta e^{2\delta t} g^{ij}u_i u_j + e^{2\delta t} \frac{d}{dt}(g^{ij}u_i u_j) \\ &\leq C g^{ij}u_i u_j + C e^{(2\delta-\gamma)t} g^{ij}u_i u_j + C e^{2\delta t} |u_i E^i|. \end{aligned} \quad (186)$$

Now since g is a scalar product: $|u_i E^i| \leq (u^i u_i)^{\frac{1}{2}} (E^i E_i)^{\frac{1}{2}}$. so, by (132) we have $|u_i E^i| \leq C(g^{ij} u_i u_j)^{\frac{1}{2}} e^{-\gamma t}$; (186) then gives:

$$\frac{dW}{dt} \leq C e^{-2\delta t} W + C e^{-\gamma t} W + C e^{(\delta-\gamma)t} W^{\frac{1}{2}} \quad (187)$$

(187) gives, since $0 < \gamma < \delta$:

$$\frac{dW}{dt} \leq C e^{-\gamma t} W + C e^{\delta t} W^{\frac{1}{2}}. \quad (188)$$

But it is well known that by (188) we have:

$$W(t) \leq z(t) \quad (189)$$

where

$$\begin{cases} \frac{dz}{dt} = C e^{-\gamma t} z + C e^{\delta t} z^{\frac{1}{2}} \\ z(0) = W(0). \end{cases} \quad (190)$$

But (190) is a Bernoulli equation whose solution is:

$$z(t) = \exp\left(-2 \int_0^t a(s) ds\right) \left[(W(0))^{\frac{1}{2}} + \int_0^t b(s) (\exp \int_0^s a(\tau) d\tau) ds \right]^2 \quad (191)$$

where:

$$a(t) = -\frac{C}{2} e^{-\gamma t}; \quad b(t) = \frac{C}{2} e^{\delta t}. \quad (192)$$

One deduces easily from (191), (192) that:

$$z(t) \leq C e^{2\delta t},$$

where $C > 0$ is a constant. Then using expression (185) of W and (189), we obtain:

$$g_{ij} u^i u^j \leq C.$$

This completes the proof of Theorem 5.1. \square

6 Energy conditions

In this section we prove that the global solution satisfies the weak and the dominant energy conditions and, under some hypothesis, the strong energy condition. Recall that a viable physical theory is supposed to fulfill at least one of the energy conditions (Hawking[9]). In fact notice that considering the stress-energy-matter tensor of the Einstein equations (2), and keeping the (\sim) to avoid any confusion, the quantity

$$(\tilde{T}_{\alpha\beta} + \tilde{\tau}_{\alpha\beta} + \rho \tilde{u}_\alpha \tilde{u}_\beta) \tilde{V}^\alpha \tilde{V}^\beta$$

represents physically, the energy density of the charged particle, measured by an observer whose velocity is \tilde{V}^α and so must be non-negative, \tilde{V}^α being a future pointing time-like vector, see [27].

We recall below the three types of energy conditions: let (\tilde{V}^α) and (\tilde{W}^α) be any two *future pointing time-like vectors*. The solution is said to satisfy:

1) the weak energy condition if:

$$(\tilde{T}_{\alpha\beta} + \tilde{\tau}_{\alpha\beta} + \rho \tilde{u}_\alpha \tilde{u}_\beta) \tilde{V}^\alpha \tilde{V}^\beta \geq 0. \quad (193)$$

2) the strong energy condition if:

$$\tilde{R}_{\alpha\beta} \tilde{V}^\alpha \tilde{V}^\beta \geq 0. \quad (194)$$

3) the dominant energy condition if:

$$(\tilde{T}_{\alpha\beta} + \tilde{\tau}_{\alpha\beta} + \rho \tilde{u}_\alpha \tilde{u}_\beta) \tilde{V}^\alpha \tilde{W}^\beta \geq 0. \quad (195)$$

Obviously, (195) implies (193), just setting: $\tilde{W}^\alpha = \tilde{V}^\alpha$.

We begin by proving:

Proposition 6.1.

Let \tilde{V}^α and \tilde{W}^α be two future pointing time-like or null vectors. Then

$$\tilde{V}^\alpha \tilde{W}_\alpha \leq 0 \quad (196)$$

Proof.

Since $\tilde{V}^\alpha \tilde{W}_\alpha = \tilde{g}_{\alpha\beta} \tilde{V}^\alpha \tilde{W}^\beta$ and given the definition (1) of \tilde{g} , (196) is equivalent to:

$$-\tilde{V}^0 \tilde{W}^0 + g_{ij} \tilde{V}^i \tilde{W}^j \leq 0. \quad (197)$$

Now, \tilde{V}^α , \tilde{W}_α being future pointing time-like or null we have:

$$\tilde{V}^\alpha \tilde{V}_\alpha \leq 0; \quad \tilde{V}^0 \geq 0; \quad \tilde{W}^\alpha \tilde{W}_\alpha \leq 0; \quad \tilde{W}^0 \geq 0$$

,or, equivalently:

$$0 \leq g_{ij} \tilde{V}^i \tilde{V}^j \leq (\tilde{V}^0)^2; \quad \tilde{V}^0 \geq 0; \quad 0 \leq g_{ij} \tilde{W}^i \tilde{W}^j \leq (\tilde{W}^0)^2; \quad \tilde{W}^0 \geq 0;$$

hence:

$$\begin{cases} 0 \leq (g_{ij} \tilde{V}^i \tilde{V}^j)^{\frac{1}{2}} \leq \tilde{V}^0 \\ 0 \leq (g_{ij} \tilde{W}^i \tilde{W}^j)^{\frac{1}{2}} \leq \tilde{W}^0 \end{cases}$$

which gives:

$$0 \leq (g_{ij}\tilde{V}^i\tilde{V}^j)^{\frac{1}{2}}(g_{ij}\tilde{W}^i\tilde{W}^j)^{\frac{1}{2}} \leq \tilde{V}^0\tilde{W}^0. \quad (198)$$

But since g is a scalar product, we have:

$$|g_{ij}\tilde{V}^i\tilde{W}^j| \leq (g_{ij}\tilde{V}^i\tilde{V}^j)^{\frac{1}{2}}(g_{ij}\tilde{W}^i\tilde{W}^j)^{\frac{1}{2}} \quad (199)$$

(197) then follows from (198) and (199). \square

Next we prove this important result for the Maxwell tensor $\tilde{\tau}_{\alpha\beta}$ defined by (6):

Proposition 6.2.

For any two future pointing time-like vectors $(\tilde{V}^\alpha), (\tilde{W}^\alpha)$, we have:

$$\tilde{\tau}_{\alpha\beta}\tilde{V}^\alpha\tilde{W}^\beta \geq 0. \quad (200)$$

Proof.

It will be enough if we could prove (200) by choosing any suitable frame. Let us consider the frame of the four vectors:

$l = (l^\alpha); n = (n^\alpha); x = (x^\alpha); y = (y^\alpha)$, satisfying the following properties:

$$l_\alpha l^\alpha = n_\alpha n^\alpha = l_\alpha x^\alpha = l_\alpha y^\alpha = n_\alpha x^\alpha = n_\alpha y^\alpha = 0. \quad (201)$$

Now inspired for instance by the case where the electromagnetic fields $\tilde{F}_{\alpha\beta}$ derives from a potential vector, the antisymmetric 2-form $\tilde{F}_{\alpha\beta}$ can be written in one of the two following general forms:

$$\tilde{F}_{\alpha\beta} = \frac{A}{2}(l_\alpha n_\beta - l_\beta n_\alpha) + \frac{B}{2}(x_\alpha y_\beta - x_\beta y_\alpha) \quad (202)$$

or:

$$\tilde{F}_{\alpha\beta} = \frac{C}{2}(l_\alpha x_\beta - l_\beta x_\alpha) \quad (203)$$

where A, B, C are constants. In fact, since $M = \mathbb{R} \times G$, with G is a *simply connected* Lie group, by Poincare Lemma, $\tilde{F}_{\alpha\beta}$ is an exact form. Then, there exists a potential vector \tilde{A}_α over M such that: $\tilde{F}_{\alpha\beta} = \tilde{\nabla}_\alpha \tilde{A}_\beta - \tilde{\nabla}_\beta \tilde{A}_\alpha$ then formula (202) generalizes the form given by the development of the above expression of $\tilde{F}_{\alpha\beta}$ by applying (21) in the case of the frame (e_α) whereas (203) corresponds to the case of the natural frame (∂_α) .

- Next, it shows useful to choose the constants A, B, C by assuming that we have in addition:

$$l_\alpha n^\alpha = -1; \quad x_\alpha x^\alpha = y_\alpha y^\alpha = 1; \quad x_\alpha y^\alpha = 0 \quad (204)$$

- Now consider the Maxwell tensor (6) i.e

$$\tilde{\tau}_{\alpha\beta} = -\frac{1}{4}\tilde{g}_{\alpha\beta}\tilde{F}^{\lambda\mu}\tilde{F}_{\lambda\mu} + \tilde{F}_{\alpha\lambda}\tilde{F}_\beta{}^\lambda \quad (205)$$

1°) Consider the form (202).

A direct calculation using (201), (204) gives:

$$\tilde{F}^{\lambda\mu}\tilde{F}_{\lambda\mu} = \frac{B^2 - A^2}{2}; \quad \tilde{F}_{\alpha\lambda}\tilde{F}_\beta{}^\lambda = \frac{A^2}{4}(l_\alpha n_\beta + n_\alpha l_\beta) + \frac{B^2}{4}(x_\alpha x_\beta + y_\alpha y_\beta) \quad (206)$$

so that, in this case, (205) and (206) give:

$$\tilde{\tau}_{\alpha\beta} = \frac{1}{4}\left[A^2(l_\alpha n_\beta + n_\alpha l_\beta) + B^2(x_\alpha x_\beta + y_\alpha y_\beta) + \tilde{g}_{\alpha\beta}\left(\frac{A^2 - B^2}{2}\right)\right]. \quad (207)$$

Now if we express the vectors $\tilde{V} = (\tilde{V}^\alpha)$, $\tilde{W} = (\tilde{W}^\alpha)$ in the frame (l, n, x, y) by:

$$\begin{cases} \tilde{V}^\alpha = Ml^\alpha + Nn^\alpha + Px^\alpha + Qy^\alpha \\ \tilde{W}^\alpha = M'l^\alpha + N'n^\alpha + P'x^\alpha + Q'y^\alpha \end{cases} \quad (208)$$

$$\quad (209)$$

and if we set:

$$\tilde{h}_{\alpha\beta} = -l_\alpha n_\beta - n_\alpha l_\beta + x_\alpha x_\beta + y_\alpha y_\beta \quad (210)$$

then a direct calculation, using (201) and (204) gives:

$$\tilde{h}_{\alpha\beta}\tilde{V}^\alpha\tilde{W}^\beta = \tilde{g}_{\alpha\beta}\tilde{V}^\alpha\tilde{W}^\beta = (-NM' - MN' + PP' + QQ').$$

Hence $\tilde{g}_{\alpha\beta} = \tilde{h}_{\alpha\beta}$. So, we can express $\tilde{g}_{\alpha\beta}$ in (207) by (210) and this gives:

$$\tilde{\tau}_{\alpha\beta} = \frac{A^2 + B^2}{8}\tilde{\tau}_{\alpha\beta} \quad (211)$$

where

$$\tilde{\tau}_{\alpha\beta} = l_\alpha n_\beta + n_\alpha l_\beta + x_\alpha x_\beta + y_\alpha y_\beta \quad (212)$$

- Now a direct calculation using (208), (209), (212), (201) and (204) gives:

$$\tilde{\tau}_{\alpha\beta}\tilde{V}^\alpha\tilde{W}^\beta = NM' + MN' + PP' + QQ'. \quad (213)$$

But if $\tilde{V} = (\tilde{V}^\alpha)$ and $\tilde{W} = (\tilde{W}^\alpha)$ are future pointing time-like vectors, in (208) and (209), we add:

$$\tilde{V}^\alpha\tilde{V}_\alpha \leq 0; \quad \tilde{W}^\alpha\tilde{W}_\alpha \leq 0; \quad M > 0; \quad M' > 0. \quad (214)$$

Now (214) writes, using (208), (209), (201) and (204):

$$-2MN + P^2 + Q^2 \leq 0; \quad -2M'N' + P'^2 + Q'^2 \leq 0; \quad M > 0; \quad M' > 0;$$

so we have: $P^2 + Q^2 \leq 2MN$; $P'^2 + Q'^2 \leq 2M'N'$; $M > 0$; $M' > 0$. But this implies since $M > 0$; $M' > 0$:

$$N \geq 0; N' \geq 0; MN \geq \frac{P^2 + Q^2}{2}; M'N' \geq \frac{P'^2 + Q'^2}{2}. \quad (215)$$

Then, since $M > 0$; $M' > 0$, (215) gives:

$$NM' \geq \frac{M'}{2M}(P^2 + Q^2); N'M \geq \frac{M}{2M'}(P'^2 + Q'^2); \quad (216)$$

So if we consider (213) in which $\tilde{V} = (\tilde{V}^\alpha)$ and $\tilde{W} = (\tilde{V}^\alpha)$ are future pointing, we deduce from (216) that:

$$\tilde{\tau}_{\alpha\beta} \tilde{V}^\alpha \tilde{W}^\beta \geq \frac{1}{2MM'} [M'^2(P^2 + Q^2) + M^2(P'^2 + Q'^2) + 2MM'(PP' + QQ')]. \quad (217)$$

Considering the term in the square bracket in the r.h.s of (217) as a quadratic polynomial in M, its discriminant is:

$$\Delta = M'^2[(PP' + QQ')^2 - (P^2 + Q^2)(P'^2 + Q'^2)].$$

But by the properties of the usual scalar product in \mathbb{R}^2 :

$$(PP' + QQ')^2 \leq (P^2 + Q^2)(P'^2 + Q'^2)$$

then $\Delta \leq 0$ and the r.h.s of (217) remains positive, and so is the l.h.s. Then in this case, (200) follows from (211).

2°) Consider the form (203).

A direct calculation using (201) and (204) gives this time:

$$\tilde{F}^{\lambda\mu} \tilde{F}_{\lambda\mu} = 0; \quad \tilde{F}_{\alpha\lambda} \tilde{F}_\beta{}^\lambda = \frac{C^2}{4} l_\alpha l_\beta.$$

Then (205) gives

$$\tilde{\tau}_{\alpha\beta} = \frac{C^2}{4} l_\alpha l_\beta \quad (218)$$

so if $\tilde{V} = (\tilde{V}^\alpha)$ and $\tilde{W} = (\tilde{V}^\alpha)$ are two future pointing time-like vectors, using the decomposition (208) and (209), we obtain from (201) and (204):

$$\tilde{\tau}_{\alpha\beta} \tilde{V}^\alpha \tilde{W}^\beta = \frac{C^2}{4} (l_\alpha \tilde{V}^\alpha) (l_\beta \tilde{W}^\beta) = \frac{C^2}{4} (-N)(-N') = \frac{C^2}{4} NN'. \quad (219)$$

But (215) which holds since \tilde{V} and \tilde{W} are future pointing, gives $N \geq 0$ and $N' \geq 0$. Hence, by (219) $\tilde{\tau}_{\alpha\beta} \tilde{V}^\alpha \tilde{W}^\beta \geq 0$. This completes the proof of Proposition 6.2. \square

Theorem 6.3.

The global solution of the coupled Einstein-Maxwell-Scalar Field satisfies:

1°) the weak and the dominant energy conditions.

2°) the strong energy condition if $\Lambda \geq \frac{(H(0))^2}{2}$.

Proof.

1°) we first prove that the solution satisfies the dominant energy condition (195). Let $\tilde{V} = (\tilde{V}^\alpha)$ and $\tilde{W} = (\tilde{W}^\alpha)$ be two future pointing time-like vectors. By (200) we have

$$\tilde{\tau}_{\alpha\beta} \tilde{V}^\alpha \tilde{W}^\beta \geq 0. \quad (220)$$

Next we have, since $\tilde{u} = (\tilde{u}^\alpha)$ is a time-like future pointing vector and using (196):

$$(\rho \tilde{u}_\alpha \tilde{u}_\beta) (\tilde{V}^\alpha \tilde{W}^\beta) = \rho (\tilde{u}_\alpha \tilde{V}^\alpha) (\tilde{u}_\beta \tilde{W}^\beta) \geq 0;$$

so:

$$(\rho \tilde{u}_\alpha \tilde{u}_\beta) (\tilde{V}^\alpha \tilde{W}^\beta) \geq 0. \quad (221)$$

Now the expression (23) of $\tilde{T}_{\alpha\beta}$ gives:

$$\begin{aligned} \tilde{T}_{\alpha\beta} \tilde{V}^\alpha \tilde{W}^\beta &= \tilde{T}_{00} \tilde{V}^0 \tilde{W}^0 + \tilde{T}_{ij} \tilde{V}^i \tilde{W}^j \\ &= \frac{1}{2} (\dot{\Phi}^2 + m^2 \Phi^2) \tilde{V}^0 \tilde{W}^0 + \frac{1}{2} (\dot{\Phi}^2 - m^2 \Phi^2) g_{ij} \tilde{V}^i \tilde{W}^j \end{aligned}$$

then:

$$\tilde{T}_{\alpha\beta} \tilde{V}^\alpha \tilde{W}^\beta = \frac{1}{2} \dot{\Phi}^2 (\tilde{V}^0 \tilde{W}^0 + g_{ij} \tilde{V}^i \tilde{W}^j) + \frac{1}{2} m^2 \Phi^2 (\tilde{V}^0 \tilde{W}^0 - g_{ij} \tilde{V}^i \tilde{W}^j). \quad (222)$$

But by (196) which is equivalent to (197), the last term in the r.h.s of (222) is positive.

Next, since g is a scalar product and \tilde{V} , \tilde{W} are future pointing vectors, we deduce from (198) and (199), that

$$|g_{ij} \tilde{V}^i \tilde{W}^j| \leq \tilde{V}^0 \tilde{W}^0;$$

then

$$g_{ij} \tilde{V}^i \tilde{W}^j \geq -\tilde{V}^0 \tilde{W}^0. \quad (223)$$

(223) then shows that the first term in the r.h.s of (222) is also positive. Consequently,

$$\tilde{T}_{\alpha\beta} \tilde{V}^\alpha \tilde{W}^\beta \geq 0 \quad (224)$$

(195) then follows from (220), (221) and (224). Hence the dominant energy condition (195) is satisfied.

- Setting in (195), $\widetilde{W} = \widetilde{V}$, we have (223). Hence the weak energy condition (193) is satisfied.

2°) We now prove the strong energy condition (194).

- Let $\widetilde{V} = (\widetilde{V}^\alpha)$ a future pointing time-like vector. We deduce from the Einstein equations (2) that:

$$\widetilde{R}_{\alpha\beta} \widetilde{V}^\alpha \widetilde{V}^\beta = \left(\frac{1}{2}\widetilde{R} - \Lambda\right) \widetilde{V}^\alpha \widetilde{V}_\alpha + 8\pi(\widetilde{T}_{\alpha\beta} + \widetilde{\tau}_{\alpha\beta} + \rho \widetilde{u}_\alpha \widetilde{u}_\beta) \widetilde{V}^\alpha \widetilde{V}^\beta. \quad (225)$$

Since (193) is satisfied, the second term in the r.h.s of (225) is positive. In the first we have :

$$\widetilde{R} = \widetilde{g}^{\alpha\beta} \widetilde{R}_{\alpha\beta} = \widetilde{g}^{00} \widetilde{R}_{00} + g^{ij} \widetilde{R}_{ij} = -\widetilde{R}_{00} + g^{ij} \widetilde{R}_{ij}. \quad (226)$$

Recall the classical formula linking \widetilde{R}_{ij} and R_{ij} :

$$\widetilde{R}_{ij} = R_{ij} - \partial_t k_{ij} + H k_{ij} - 2k_{il} k_j^l.$$

Contracting by g^{ij} yields:

$$g^{ij} \widetilde{R}_{ij} = R - g^{ij} \partial_t k_{ij} + H^2 - 2k_{ij} k^{ij} \quad (227)$$

write: $g^{ij} \partial_t k_{ij} = \partial_t (g^{ij} k_{ij}) - k_{ij} \partial_t g^{ij} = \partial_t H - 2k_{ij} k^{ij}$, (227) then gives:

$$g^{ij} \widetilde{R}_{ij} = R - \partial_t H + H^2. \quad (228)$$

Now setting in the Einstein equations $\alpha = \beta = 0$ yields:

$$\widetilde{R}_{00} = -\frac{\widetilde{R}}{2} + \Lambda + 8\pi(\widetilde{T}_{00} + \widetilde{\tau}_{00} + \rho \widetilde{u}_0^2). \quad (229)$$

(226) gives, using (227) using (229):

$$\widetilde{R} = \frac{\widetilde{R}}{2} - \Lambda - 8\pi(\widetilde{T}_{00} + \widetilde{\tau}_{00} + \rho \widetilde{u}_0^2) + R - \partial_t H + H^2.$$

From where we deduce, using $\widetilde{T}_{00} + \widetilde{\tau}_{00} + \rho \widetilde{u}_0^2 \geq 0$, $R < 0$, $-\partial_t H \leq 0$, (given by (84)):

$$\frac{\widetilde{R}}{2} - \Lambda \leq -2\Lambda + H^2. \quad (230)$$

But by (77) we have: $H^2 \leq (H(0))^2$. Hence (230) gives:

$$\frac{\widetilde{R}}{2} - \Lambda \leq -2\Lambda + (H(0))^2 \quad (231)$$

but by hypothesis: $-2\Lambda + (H(0))^2 \leq 0$; hence:

$$\frac{\widetilde{R}}{2} - \Lambda \leq 0; \quad (232)$$

since $\widetilde{V}^\alpha \widetilde{V}_\alpha < 0$, (232) implies that the first term in the r.h.s of (225) is positive; we conclude that; we have: $\widetilde{R}_{\alpha\beta} \widetilde{V}^\alpha \widetilde{V}^\beta \geq 0$. This completes the proof of Theorem 6.3.

□

Concluding Remarks

In our future investigations, we will take into account the aspect "distribution", of the charged particles. For this purpose we will couple the Vlasov (resp. Boltzmann) equation in the collisionless(resp.collisional) case.

7 Appendices

A₁. Proof of formula (22)

We use the Codazzi equations which write:

$$\widetilde{R}_{\lambda i, j l} \widetilde{n}^\lambda = -\nabla_l k_{ij} + \nabla_j k_{il}. \quad (233)$$

But $\widetilde{n} = (1, 0, 0, 0)$ so (233) gives:

$$\widetilde{R}_{0i, j l} = -\nabla_l k_{ij} + \nabla_j k_{il}. \quad (234)$$

Now the curvature tensor, on (M, \widetilde{g}) writes, see [2], p.240:

$$\widetilde{R}^\lambda_{\alpha, \beta \mu} = \widetilde{e}_\beta(\gamma^\lambda_{\alpha \mu}) - \widetilde{e}_\mu(\gamma^\lambda_{\beta \alpha}) + (\gamma^l_{\beta \nu} \gamma^\nu_{\mu \alpha}) - (\gamma^\lambda_{\mu \nu} \gamma^\nu_{\beta \alpha}) - \widetilde{C}^\nu_{\beta \mu} \gamma^\lambda_{\nu \alpha}. \quad (235)$$

In particular, taking in (235): $\lambda = l$; $\alpha = j$; $\beta = i$; $\mu = 0$, we obtain, using (21), (12), (13), (18) and (19):

$$\widetilde{R}^l_{j, i 0} = -\frac{d}{dt}(\gamma^l_{ij}) - \nabla_i k^l_j. \quad (236)$$

Now (234) gives, using the symmetry properties of $\widetilde{R}^\lambda_{\alpha, \beta \mu}$:

$$\widetilde{R}^l_{j, i 0} = -\nabla^l k_{ij} + \nabla_j k^l_i. \quad (237)$$

Equalize the two values of $\widetilde{R}^l_{j, i 0}$ provided by (236), (237) to obtain (21).

A₂. Proof of Lemma 2.1

1°) Proof of (63).

First deduce from the evolution (49) in g_{ij} ; using $g^{il} g_{jl} = \delta^i_j$ that:

$$\frac{dg^{ij}}{dt} = 2k^{ij}. \quad (238)$$

Now since $H = g^{ij} k_{ij}$, (238) gives:

$$\frac{dH}{dt} = g^{ij} \frac{dk_{ij}}{dt} + 2k_{ij} k^{ij}. \quad (239)$$

A direct calculation using the evolution equation (50) in k_{ij} and the relation $\widetilde{g}^{\alpha\beta} \widetilde{\tau}_{\alpha\beta} = 0$ which implies $g^{ij} \tau_{ij} = \tau_{00}$, gives

$$g^{ij} \frac{dk_{ij}}{dt} = R + H^2 - 2k_{ij} k^{ij} + 4\pi g^{ij} (T_{ij} + \rho u_i u_j) - 12\pi (T_{00} + \rho u_0^2) - 8\pi \tau_{00} - 3\Lambda. \quad (240)$$

then (63) follows from (238) and (240)

2°) Proof of (64)

We have $\frac{dH^2}{dt} = 2H \frac{dH}{dt}$; then use (22) to obtain (64).

3°) Proof of (65)

We have

$$\frac{d(k_{ij}k^{ij})}{dt} = k_{ij} \frac{dk^{ij}}{dt} + k^{ij} \frac{dk_{ij}}{dt}. \quad (241)$$

We also have: $k^{ij} = g^{il}g^{jm}k_{lm}$; so, we have, using (238)

$$\frac{dk^{ij}}{dt} = 4k^{il}k_l^j + g^{il}g^{jm} \frac{dk_{lm}}{dt} \quad (242)$$

(242) gives by a direct calculation, using the evolution equation (50) for k_{lm} :

$$\begin{aligned} \frac{dk^{ij}}{dt} &= R^{ij} + Hk^{ij} + 2k_l^j k^{il} - 8\pi(T^{ij} + \tau^{ij} + \rho u^i u^j) \\ &+ 4\pi g^{ij} \left[-T_{00} - \rho u_0^2 + g^{lm}(T_{lm} + \rho u_l u_m) \right] - \Lambda g^{ij}. \end{aligned} \quad (243)$$

(65) then follows from (241), (243), using once more the evolution equation (50) for k_{ij} to express the last term in (241).

4°) Proof of (66) and (67)

We use the conservation laws:

$$\widetilde{\nabla}_\alpha (\widetilde{T}^{\alpha\beta} + \widetilde{\tau}^{\alpha\beta} + \rho \widetilde{u}^\alpha \widetilde{u}^\beta) = 0 \quad (244)$$

(244) writes, using the formulae (21)

$$\widetilde{e}_\alpha (\widetilde{T}^{\alpha\beta} + \widetilde{\tau}^{\alpha\beta} + \rho \widetilde{u}^\alpha \widetilde{u}^\beta) + \widetilde{\gamma}_{\alpha\lambda}^\alpha (\widetilde{T}^{\lambda\beta} + \widetilde{\tau}^{\lambda\beta} + \rho \widetilde{u}^\lambda \widetilde{u}^\beta) + \widetilde{\gamma}_{\alpha\lambda}^\beta (\widetilde{T}^{\alpha\lambda} + \widetilde{\tau}^{\alpha\lambda} + \rho \widetilde{u}^\alpha \widetilde{u}^\lambda) = 0. \quad (245)$$

It shows useful to write (243) in the form:

$$\begin{aligned} &\widetilde{e}_0 (\widetilde{T}^{0\beta} + \widetilde{\tau}^{0\beta} + \rho \widetilde{u}^0 \widetilde{u}^\beta) + e_l (\widetilde{T}^{l\beta} + \widetilde{\tau}^{l\beta} + \rho u^l \widetilde{u}^\beta) + \widetilde{\gamma}_{\alpha 0}^\alpha (\widetilde{T}^{0\beta} + \widetilde{\tau}^{0\beta} + \rho \widetilde{u}^0 \widetilde{u}^\beta) \\ &+ \widetilde{\gamma}_{\alpha i}^\alpha (\widetilde{T}^{i\beta} + \widetilde{\tau}^{i\beta} + \rho \widetilde{u}^i \widetilde{u}^\beta) + \widetilde{\gamma}_{\alpha 0}^\beta (\widetilde{T}^{\alpha 0} + \widetilde{\tau}^{\alpha 0} + \rho \widetilde{u}^\alpha \widetilde{u}^0) + \widetilde{\gamma}_{\alpha j}^\beta (\widetilde{T}^{\alpha j} + \widetilde{\tau}^{\alpha j} + \rho \widetilde{u}^\alpha \widetilde{u}^j) = 0. \end{aligned} \quad (246)$$

Then: set in (246) $\beta = 0$ and use (19) to obtain (66).

set in (246) $\beta = j$ and use (19) to obtain (67).

5°) Proof of (68)

We have $R = g^{ij}R_{ij}$, where R_{ij} given by (58), then:

$$\frac{dR}{dt} = R_{ij} \frac{dg^{ij}}{dt} + g^{ij} \frac{dR_{ij}}{dt}.$$

So we have, using (238)

$$\frac{dR}{dt} = 2R_{ij}k^{ij} + g^{ij} \frac{dR_{ij}}{dt}. \quad (247)$$

Now we verifies by a direct calculation, using the formula (21) and the expression (58) of R_{ij} that:

$$\frac{dR_{ij}}{dt} = \nabla_l \left(\frac{d\gamma_{ij}^l}{dt} \right) \quad (248)$$

(248) gives, using formula (22) and since $H = g^{ij}k_{ij}$ depends only on t :

$$g^{ij} \frac{dR_{ij}}{dt} = -2\nabla_l \nabla_i k^{il} \quad (249)$$

(68) then follows from (247) and (249).

A₃. Proof of (70)

It is easily seen, using the definition of A_j in Lemma 2.2, that

$$A_l = g_{jl}B^j \quad (250)$$

where:

$$B^j = \nabla_i k^{ij} - 8\pi(T^{0j} + \tau^{0j} + \rho u^0 u^j). \quad (251)$$

We then have, differentiating (250) and using equation (49) in g_{ij} :

$$\frac{dA_l}{dt} = -2k_{jl}B^j + g_{jl} \frac{dB^j}{dt}. \quad (252)$$

Now we have by (251):

$$\frac{dB^j}{dt} = \frac{d}{dt}(\nabla_i k^{ij}) - 8\pi \frac{d}{dt}(T^{0j} + \tau^{0j} + \rho u^0 u^j) \quad (253)$$

But $\nabla_i k^{ij} = \partial_i k^{ij} + \gamma_{il}^i k^{lj} + \gamma_{il}^j k^{il}$; then we have:

$$\frac{d(\nabla_i k^{ij})}{dt} = \partial_i \left(\frac{dk^{ij}}{dt} \right) + \gamma_{il}^i \frac{dk^{lj}}{dt} + \gamma_{il}^j \left(\frac{dk^{il}}{dt} \right) + \left(\frac{d\gamma_{il}^i}{dt} \right) k^{lj} + \left(\frac{d\gamma_{il}^j}{dt} \right) k^{il}. \quad (254)$$

Now use (243) to express $\frac{dk^{ij}}{dt}$, (22) to express $\frac{d\gamma_{il}^j}{dt}$ and obtain:

$$\begin{aligned} \frac{d(\nabla_i k^{ij})}{dt} &= \nabla_i R^{ij} + H \nabla_i k^{ij} + 2(\nabla_i k^{il})k_l^j + 2k^{il} \nabla_i k_l^j - 8\pi \nabla_i (T^{ij} + \tau^{ij} + \rho u^i u^j) \\ &+ (\nabla^j k_{il} - \nabla_i k_l^j + \nabla_l k_i^j) k^{il} \end{aligned} \quad (255)$$

(253) then gives using (255) to express the first term, (67) to express the second term, and taking into account the expression (251) of B^j .

$$\frac{dB^j}{dt} = HB^j + 2k_i^j B^i + \nabla_i R^{ij} \quad (256)$$

But by the Bianchi identities:

$$\nabla_i R^{ij} = \frac{1}{2} \nabla^i R = g^{ij} \nabla_j R = 0$$

since R depends only on t .

Finally (70) follows from (252), (256) and (250)

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